PAPER 14 NUMERICAL METHODS

Unit I Solution of algebraic and transcendental equations

Iteration method, bisection method, Newton – Raphson method, rate of convergence – solution of polynomial equations – Brige Vieta method – Bairstow method.

Unit II Solution of simultaneous equations

Direct method – Gauss elimination method – Gauss-Jordan method – iterative methods – Gauss seidal iterative method – Eigen values and Eigen vectors of matrices – Jacobi method for symmetric matrices.

Unit III Interpolation

Interpolation formula for unequal intervals – Lagrange's method – Interpolation formula for equal intervals – Newton's forward interpolation formula – Newton's Backward interpolation formula - least squares approximation method.

Unit IV Numerical differentiation and integration

Methods based on interpolation – Newton's forward difference formula – Newton's backward formula – numerical integration – Quadrature formula (Newton's cote's formula) – Trapezoidal rule, Simpson's $1/3^{rd}$ rule, $3/8^{th}$ rule – Gauss quadrature formula – Gauss two point formula and three point formula.

Unit V Initial value problems

Solution of first order differential equations – Taylor series method, Euler's method, Runge-Kutta methods (fourth order) – Milne's predictor – corrector method – Adam-Moulton method.

Books for study and Reference

1. Numerical methods for scientific and engineering computations – Jain and Iyengar.
2. Numerical methods – Venkatraman.
3. Numerical methods – Sastry.
4. Numerical methods – A. Singaravelu.

Unit I - Solution of algebraic and transcendental equations

An equation f(x)=0 which is only a polynomial in x is known as an algebraic equation.

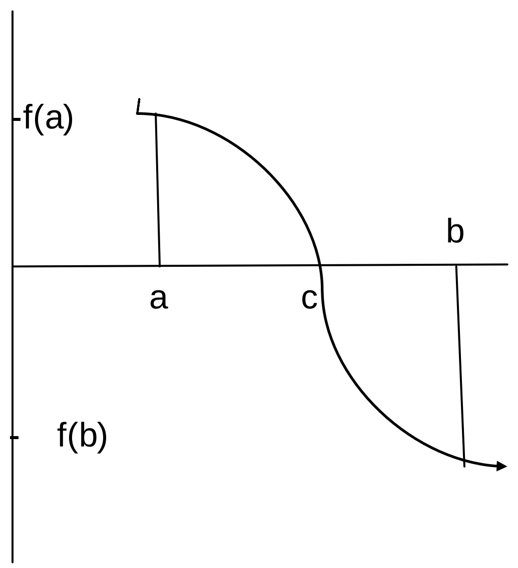
eg. $X^5 - x^4 + x - 26 = 0$.

Whereas, an equation containing transcendental terms, such as exponential terms, trigonometric terms, logarithmic terms etc, are known as transcendental equations.

eg. $e^x - x + 5 = o$; $x + 5 \cos x + 2 = 0$

The point at which the function y = f (x) cuts in the x – axis is known as the root of the equation (or) zero of the equation.

y = f (x)



y = f (a) is positive

y = f (b) is negative

y = f (c) is zero
Therefore c is the root of the equaton

Two points a and b are to be identified for the given equation, such that for one of the points a, y=f(a) is positive and for the point b, f(b) is negative. Then the first approximate root is given be $x_0 = (a + b) /2$.

Iteration Method :

The root of the given equation $f(x) = 0$ can be determined by iteration method. Let x_0 be the approximate initial root of the given equation calculated as mentioned above. We have to rewrite the given equation in the form of

$$x = \Phi(x)$$

The first approximate root is given by $x_1 = \Phi(x_0)$

The successive approximations are given by

$x_2 = \Phi(x_1)$

$x_3 = \Phi(x_2)$

…………..

$x_n = \Phi(x_{n-1})$

The process of finding root is continued until the successive roots are same to the required accuracy.

Note

The iteration method will give converging results if

$\Phi'(x) < 1.$

e.g. Consider the equation $f(x) = x^2 + x - 3 = 0$

$f(0) = -3$, negative; $f(1) = -1$, negative; $f(2) = 3$, positive

Hence the root lies between 1 and 2. The initial root is calculated as

$x_0 = (1 + 2) / 2 = 1.5$

From the given equation, we can write $x = 3 - x^2 = \Phi(x)$.

Differentiating, we get $\Phi'(x) = -2(x)$

$|\Phi'(1.5)| = 3 > 1$

Therefore this form of Φ(x) will not give converging solutions.

If we rewrite the given equation as $x(x + 1) = 3$; then we can write

$x = 3 / (x + 1) = \Phi(x)$

$\Phi'(x) = -3/(1 + x)^2$ and $\Phi'(1.5) < 1$

Therefore this form of Φ (x) will give converging solutions.

Problems:

1. **Find the root of the equation $x^3 + x^2 - 1 = 0$ by iteration method correct to two decimal places.**

Given $f(x) = x^3 + x^2 - 1 = 0$

$f(0) = -1$ negative

$f(1) = 1$, positive

Therefore the initial root is $x_0 = (0 + 1) / 2 = 0.5$

We can rewrite the given equation $x^3 + x^2 - 1 = 0$ as

$x^2 (x + 1) = 1$

$x = 1/\sqrt{(1 + x)}$

Therefore $\Phi(x) = 1/\sqrt{(1 + x)}$ and $\Phi'(x) = -1/2(x + 1)^{3/2}$

$|\Phi'(x)|_{x = 0.5} < 1$

Therefore this form of Φ(x) will give converging solutions.

The first approximate solution is

$x_1 = \Phi(x_0) = 1/\sqrt{(1 + x_0)} = 1/\sqrt{(1 + 0.5)} = 0.81649$

$x_2 = \Phi(x_1) = 1/\sqrt{(1 + 0.81649)} = 0.74196$

$x_3 = \Phi(x_2) = 1/\sqrt{(1 + 0.74196)} = 0.75767$

$x_4 = \Phi(x_3) = 1/\sqrt{(1 + 0.75767)} = 0.75427$

Since $x_3 \approx x_4$ upto two decimal places, the root of the given equation is 0.7543.

2. Find the root of the equation $3x - \log_{10} x - 6 = 0$ by iteration method.

Given $f(x) = 3x - \log_{10} x - 6$

$f(1) = -3$ (negative) $f(2) = -0.3010$ (negative)

$f(3) = 2.5229$ (positive)

The root lies between 2 and 3.

Let the initial approximate root be, $x_0 = 2$

We can rewrite the equation $3x - \log_{10} x - 6 = 0$ as

$X = 1/3(6 + \log_{10} x)$

Therefore $\Phi(x) = 1/3(6 + \log_{10} x)$

The first approximate solution is

$x_1 = \Phi(x_0) = 1/3(6 + \log_{10} x_0) = 1/3(6 + \log_{10} 2) = 2.1003$

$x_2 = \Phi(x_1) = 1/3(6 + \log_{10} 2.1003) = 2.1074$

$x_3 = \Phi(x_2) = 1/3(6 + \log_{10} 2.1074) = 2.1079$

Since $x_2 \approx x_3$ up to three decimal places, the root of the given equation is 2.1079

3. Find the root of the equation $f(x) = 3x - \cos x - 1 = 0$ by iteration method.

Given $f(x) = 3x - \cos x - 1 = 0$ Now $f(0)$ = negative and $f(1)$ = positive

Let us take initial root as $x_0 = 0.6$

We can write $x = 1/3 (1 + \cos x) = \Phi (x)$

$x_1 = \Phi (x_0) = 1/3 (1 + \cos 0.6) = 0.60845$

$x_2 = \Phi (x_1) = 1/3 (1 + \cos 0.60845) = 0.60684$

$x_3 = \Phi (x_2) = 1/3 (1 + \cos 0.60684) = 0.60715$

$x_4 = \Phi (x_3) = 1/3 (1 + \cos 0.60715) = 0.6071$

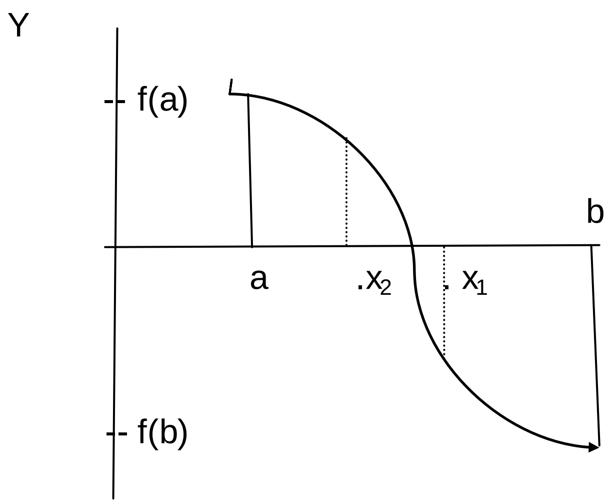
Since $x_3 \approx x_4$ correct to three decimal places, the root of the given equation is 0.6071.

Bisection method:

If f (a) and f (b) are of opposite signs, then the equation f (x) = 0 will have at least one real root between a and b. The bisection method is useful to find the root between a and b. The first approximate root is taken as the midpoint of the range a and b. i.e. $x_0 = (a + b) / 2$

Then we have to find $f(x_0)$. Let us assume f(a) as positive and f(b) as negative. Let $f (x_0)$ be negative. Then the root lies between a and x_0.

If $f (x_0)$ is positive, then the root lies between x_0 and b. We have to bisect the interval in which the root lies and the process is to be repeated.



In the above diagram, for the given curve y = f (x), f (a) is positive and f (b) is negative. The first approximate root is the midpoint of a and b

i.e. $x_1 = (a + b) / 2$

f (x_1) is negative. Therefore the next approximation is the midpoint of x_1 and a

$x_2 = (a + x_1) / 2$

Similarly $x_3 = (x_1 + x_2)/ 2$

The process is repeated until two successive roots are equal to the required degree of approximation.

Problems :

1. Find the root of the equation $x^3 - x - 1 = 0$ correct to two decimal places by bisection method.

Let $f(x) = x^3 - x - 1$

f (0) = -1 ie negative

f (1) = -1 negative

f (1.5) = 0.875 is positive

Therefore the initial root is $x_0 = (1 + 1.5) / 2 = 1.25$

$f(x_0) = f(1.25) = (1.25)^3 - 1.25 - 1 = -0.29688$

Since f (x_0) is negative, the root lies between 1.25 and 1.5. The next approximate root is

$x_1 = (1.25 + 1.5)/ 2 = 1.375$

$f(1.375) = (1.375)^3 - 1.375 - 1 = 0.22461$

Since f (1.375) is positive, the next root lies between 1.25 and 1.375

Ie. $x_2 = (1.25 + 1.375) / 2 = 1.3125$

$f(1.3125) = (1.3125)^3 - 1.3125 - 1 = -0.051514$ (negative)

Therefore the root lies between 1.3125 and 1.375

$x_3 = (1.3125 + 1.375) / 2 = 1.3438$

$f(x_3) = f(1.3438) = 0.0824$ (Positive)

Therefore the root lies between 1.3438 and 1.3125

Ie. $x_4 = (1.3125 + 1.3438) / 2 = 1.3282$

$f(x_4) = 0.014898$ (Positive)

Therefore the root lies between 1.3282 and 1.3125

Therefore $x_5 = (1.3125 + 1.3282) / 2 = 1.3204$

Since $x_4 \approx x_5$ correct to two decimal places, the root is 1.3204

Note: If $f(x) \approx 0$, then x is the root. Here $f(x_5) = 0.0183$ which is very small and near to zero. Therefore the root is 1.3204.

2. Find the root of the equation $x \log_{10} x - 1.2 = 0$ by bisection method.

Let $f(x) = x \log_{10} x - 1.2$

$f(2) = 2 \log_{10} 2 - 1.2 = -0.598$ (negative)

$f(3) = 3 \log_{10} 3 - 1.2 = 0.2313$ (positive)

Therefore the root lies between 2 and 3

$x_0 = (2 + 3) / 2 = 2.5$

$f(x_0) = f(2.5) = 2.5 \log_{10} 2.5 - 1.2 = -0.2053$

Since f (2.5) is negative and f (3) is positive the root lies between 2.5 and 3

Therefore the next approximate root is

$x_1 = (2.5 + 3) / 2 = 2.75$

$f(2.75) = 2.75 \log_{10} 2.75 - 1.2 = 0.008$

Since f (2.75) is > 0 and f (2.5) is < o the next root lies between 2.75 and 2.5

Therefore x_2 = (2.75 + 2.5) / 2 = 2.625

$f(x_2) = f(2.625) = -0.10$

Therefore the next approximation is

$x_3 = (2.625 + 2.75) / 2 = 2.6875$

$f(x_3) = f(2.6875) \approx 0$

Therefore the root of the equation is 2.6875

Newton- Raphson Method:

Let x_0 be the approximate root of the equation f (x) = 0

Let $x_1 = x_0 + h$ be the exact root of the equation

Therefore $f(x_1) = 0$

By Taylor's series expansion, we can write

$f(x_1) = f(x_0 + h) = f(x_0) + h/1!\, f'(x_0) + (h^2 / 2!)\, f''(x_0) + \ldots$

Since h is very small, h^2 is negligibly small and we can ignore the higher order terms. Therefore we can write

$f(x_1) = f(x_0) + h f'(x_0) = 0$

i.e. $h = -f(x_0) / f'(x_0)$

Therefore $x_1 = x_0 + h = x_0 - f(x_0) / f'(x_0)$

The next approximation is

$x_2 = x_1 - f(x_1) / f'(x_1)$

In general, $x_{n+1} = x_n - f(x_n) / f'(x_n)$

This is known as Newton – Raphson iteration formula.

Problems:

1. Find the root of the equation $x^3 - 3x + 1 = 0$ by Newton – Raphson method.

Given $f(x) = x^3 - 3x + 1 = 0$

$f'(x) = 3x^2 - 3$

$f(1) = 1 - 3 + 1 = -1$ (Negative)

$f(2) = 2^3 - 6 + 1 = 3$ (Positive)

Therefore $x_0 = (1 + 2) / 2 = 1.5$

$f(x_0) = 1.5^3 - 3 \times 1.5 + 1 = -0.125$

$f'(x_0) = 3(1.5)^2 - 3 = 3.75$

$x_1 = 1.5 - (-0.125/3.75) = 1.5333$

$f(x_1) = (1.5333)^3 - 3 \times 1.5333 + 1 = 0.0049$

$f'(x_1) = 3(1.5333)^2 - 3 = 4.053$

$x_2 = 1.5333 - (0.0049/4.053) = 1.5321$

Since $x_2 \approx x_3$ the root of the given equation is 1.5321

2. Find the root of the equation $\cos x - xe^x = 0$ by Newton – Raphson method.

$f(x) = \cos x - xe^x$

$f'(x) = -\sin x - xe^x - e^x$

Let $x_0 = 0.5$

$x_1 = x_0 - f(x_0) / f^!(x_0) = 0.5 – f(0.5)/ f^!(0.5)$

$= 0.5 – (0.533/-2.9525)$

$= 0.5 + 0.0181 = 0.5181$

$x_2 = x_1 - f(x_1) / f^!(x_1) = 0.5181 – f(0.5181)/ f^!(0.5181)$

$= 0.5181 – (- 0.00104 /-3.0438)$

$= 0.5181 - 0.00034 = 0.5178$

$x_3 = x_2 - f(x_2) / f^!(x_2) = 0.5178 – f(0.5178)/ f^!(0.5178)$

$= 0.5178 – (- 0.00012 /-3.0422)$

$= 0.5178$

Since $x_2 \approx x_3$ the root of the given equation is 0.5178.

Rate of convergence of Newton-Raphson method

Let α be the root of the equation f(x) = 0

Let x_n be the approximate root of the equation and e_n be the small error by which x_n and α differs

Therefore $x_n = \alpha + e_n$

Similarly $x_{n+1} = \alpha + e_{n+1}$

Newton- Raphson formula is

$$x_{n+1} = x_n – f(x_n) / f'(x_n)$$

$$\alpha + e_{n+1} = \alpha + e_n - f(\alpha + e_n) / f'(\alpha + e_n)$$

$$e_{n+1} = e_n – [f(\alpha + e_n) / f'(\alpha + e_n)]$$

using Taylor's series expansion we can write

$$e_{n+1} = e_n - \left[\frac{f(\alpha)+en\, f'(\alpha)+\frac{en^2}{2!} f''(\alpha)+ \ldots\ldots\ldots}{f'(\alpha)+ en\, f''(\alpha)+ \ldots\ldots\ldots} \right]$$

$$= e_n - \left[\frac{f(\alpha)+en\, f'(\alpha)+\frac{en^2}{2!} f''(\alpha)}{f'(\alpha)+ en\, f''(\alpha)} \right]$$

$$= \frac{e_n f^{!}(\alpha) + e_n^2 f^{!!}(\alpha) - f(\alpha) - e_n f^{!}(\alpha) - e_n^2 f^{!!}(\alpha)/2}{f^{!}(\alpha) + e_n f^{!!}(\alpha)}$$

$$= \frac{\frac{e^2_n f^{!!}(\alpha)}{2!}}{f^{!}(\alpha)\,[\,1 + e_n f^{!!}(\alpha)/ f^{!}(\alpha)]}$$

$$= K e^2_n$$

$$e_{n+1} \propto e^2_n$$

i.e. the error in successive steps decreases as the square of error in the previous step. Thus the order of convergence is two.

Birge – Vieta method :

The real root of a polynomial equation f(x) = 0 can be obtained by this method.

Let $f(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots\ldots\ldots\ldots + a_n$

Let (x – r) be a factor of f (x).

$f(x) = (x - r)\, q(x) + R$

where $q(x) = b_0 x^{n-1} + b_1 x^{n-2} + b_2 x^{n-3} + \ldots\ldots\ldots\ldots + b_{n-1}$

and R is the remainder

Let r_0 be the initial approximation to r. Using Newton – Raphson method, the close approximation to the root r is given by $r_1 = r_0 - f(r_0)/ f^{!}(r_0)$.

The values of $f(r_0)$ and $f^!(r_0)$ are calculated by synthetic division formula in Brge – Vieta method.

Substituting f (x) and q (x) in the equation f (x) = (x - r) q (x) + R and comparing the coefficients of like power of x on both sides we get

$b_0 = a_0$

$b_1 = a_1 + r_0 a_0$

$b_2 = a_2 + r_0 b_1$

$R = b_n = a_n + r_0 b_{n-1}$

In the synthetic division method,

$f(r_0) = R = b_n$ and $C_i = b_i + r_0 C_{i-1}$ where $C_0 = b_0$ and

$f^!(r_0) = dR / dr_0 = C_{n-1}$

Substituting in the formula $r_1 = r_0 – f(r_0)/ f^!(r_0)$

we get $r_1 = r_0 - b_n / C_{n-1}$

Synthetic division

	a_0	a_1	a_2	………..	a_{n-2}	a_{n-1}	an
r_0		r_0b_0	r_0b_1	………..	r_0b_{n-3}	r_0b_{n-2}	r_0b_{n-1}
	b_0	b_1	b_2	………..	b_{n-2}	b_{n-1}	$\boxed{bn}$
r_0		r_0c_0	r_0c_1	………..	r_0c_{n-3}	r_0c_{n-2}	……
	c_0	c_1	c_2	………..	c_{n-2}	$\boxed{c_{n-1}}$	……..

Using the formula $r_1 = r_0 - b_n / C_{n-1}$ the approximate root r_1 is calculated.

Replacing the value of r_1 in r_0 in the above synthetic division method the next approximated value $r_2 = r_1 - b_n / C_{n-1}$ can be obtained.

Problem:

1**. Find the root of the equation $x^4 + 2x^3 - 21x^2 - 22x + 40 = 0$ using Birge – Vieta method. Perform two iterative. Take initial root as 3.5.**

Solution:

Here	$a_0 = 1$	$a_1 = 2$	$a_2 = -21$	$a_3 = -22$	$a_4 = 40$
	$r_o = 3.5$				
	1	2	-21	-22	40
3.5		3.5	19.25	-6.125	-98.4375
	1	5.5	-1.75	-28.125	-58.4375 (= b_n)
3.5		3.5	31.5	104.125	
	1	9	29.75	76 (=c_{n-1})	

$r_1 = r_0 - b_n / C_{n-1} = 3.5 + (58.4375 / 76) = 4.2689$

Second iteration

	1	2	-21	-22	40
4.2689		4.2689	26.7613	24.5944	11.0752
	1	6.2689	5.7613	2.5944	51.0752 (= b_n)
3.5		4.2689	44.9848	216.63	
	1	10.5378	50.7461	219.224 (= c_{n-1})	

$r_2 = r_1 - b_n / C_{n-1} = 4.2689 - (51.0752 / 219.224) = 4.0359$

The root of the given equation is 4.0359

2. **Find the root of the equation $x^3 + 2x^2 + 10x - 20 = 0$ using Birge – Vieta method. Take initial root as 1. Perform two iterations.**

Solution:

Here	$a_0 = 1$	$a_1 = 2$	$a_2 = 10$	$a_3 = -20$
	$r_o = 1$			
	1	2	10	-20
1		1	3	13
	1	3	13	-7 (= b_n)
1		1	4	17
	1	4	17 (= c_{n-1})	

$r_1 = r_0 - b_n / C_{n-1} = 1 + (7 / 17) = 1.4118$

The Second iteration is

	1	2	10	-20
1.4118		1.4118	4.8168	20.9184
	1	3.4118	14.8168	0.9184 (= b_n)
1.4118		1.4118	6.8100	
	1	4.8236	21.6268 (= c_{n-1})	

$r_2 = r_1 - (b_n / C_{n-1}) = 1.4118 – (0.9184 / 21.6268) = 1.3693$

The root of the given equation is 1.3693.

Bairstow Method:

In this method, the quadratic factor of the form $x^2 + px + q$ for an n^{th} degree polynomial f (x) is obtained by iterative process.

Let $f(x) = a_0x^4 + a_1x^3 + a_2x^2 + a_3x + a_4$

be a fourth degree polynomial.

The quadratic factor be $x^2 + px + q$. The quotient will be of the form

$b_0x^2 + b_1x + b_2$ and the remainder will be $Rx + S$.

Therefore, we can write

$$a_0x^4 + a_1x^3 + a_2x^2 + a_3x + a_4 = (x^2 + px + q)(b_0x^2 + b_1x + b_2) + Rx + S.$$

Equating the coefficients of like powers of x on both sides, we can write

$b_0 = a_0$

$b_1 = a_1 - pb_0$

$b_2 = a_2 - pb_1 - qb_0$

$b_3 = a_3 - pb_2 - qb_1$

$b_4 = a_4 - pb_3 - qb_2$ in general

$b_k = a_k - pb_{k-1} - qb_{k-2}$

and we assume that $R = b_3$ and $S = b_4 + pb_3$

We can write $c_k = b_k - pc_{k-1} - qc_{k-2}$ with $c_0 = b_0$

The approximations to p and q are given by Δp and Δq.

$$\Delta p = \frac{b_3c_2 - b_4c_1}{c_2^2 - c_1(c_3 - b_3)}$$

$$\Delta q = \frac{b_4c_2 - b_3(c_3 - b_3)}{c_2^2 - c_1(c_3 - b_3)}$$

The approximated solution is given by $p_1 = p + \Delta p$ and $q_1 = q + \Delta q$

The iterations are repeated to get still accurate values.

The synthetic division procedure is given by

	a_0	a_1	a_2	a_3	a_4
-p		$-pb_0$	$-pb_1$	$-pb_2$	$-pb_3$
-q			$-qb_0$	$-qb_1$	$-qb_2$
	b_0	b_1	b_2	b_3	b_4
-p		$-pc_0$	$-pc_1$	$-pc_2$	
-q			$-qc_0$	$-qc_1$	
	c_0	c_1	c_2	c_3	

Note:

$b_0 = a_0$

$b_1 = a_1 - pb_0$ $\qquad c_0 = b_0$

$b_2 = a_2 - pb_1 - qb_0$ $\qquad c_1 = b_1 - pc_0$

$b_3 = a_3 - pb_2 - qb_1$ $\qquad c_2 = b_2 - pc_1 - qc_0$

$b_4 = a_4 - pb_3 - qb_2$ $\qquad c_3 = b_3 - pc_2 - qc_1$

Problems:

1. Find the quadratic factor of the form $x^2 + px + q$ from the polynomial $X^4 - 3x^3 - 4x^2 - 2x + 8 = 0$ by Bairstow method with initial values $p_0 = q_0 = 1.5$

Solution:

	1	-3	-4	-2	8
$-p_0 = -1.5$		-1.5	6.75	-1.875	-4.3125
$-q_0 = -1.5$			-1.5	6.75	-1.875
	1	-4.5	1.25	2.875 (=b_3)	1.8125 (=b_4)
$-p_0 = -1.5$		-1.5	9.0	-13.125	
$-q_0 = -1.5$			-1.5	9	
	1	-6.0 (c_1)	8.75 (=c_2)	-1.125 (=c_3)	

$$\Delta p_0 = \frac{b_3c_2 - b_4c_1}{c_2^2 - c_1(c_3 - b_3)}$$

$$= \frac{(2.875 \times 8.75) + (1.8125 \times 6)}{(8.75)^2 - 6(1.25 + 2.875)}$$

$$= 0.6954$$

$$\Delta q_0 = \frac{b_4c_2 - b_3(c_3 - b_3)}{c_2^2 - c_1(c_3 - b_3)}$$

$$= \frac{(1.8125 \times 8.75) + (2.875 \times 4.125)}{(8.75)^2 - 6(1.25 + 2.875)}$$

$$= 0.534982$$

Therefore

$p_1 = p_0 + \Delta p_0 = 2.1954$ and $q_1 = q_0 + \Delta q_0 = 2.0350$

Second iteration

	1	-3	-4	-2	8
$-p_1 = -2.1954$		-2.1954	11.4060	-11.7915	7.0668
$-q_1 = -2.0350$			-2.0350	10.5726	-10.930
	1	-5.1954	5.3710	-3.2189 (b_3)	4.1318 (b_4)
$-p_1 = -2.1954$		-2.1954	16.2258	-42.946	
$-q_1 = -2.0350$			-2.035	15.0402	
	1	-7.3908 (c_1)	19.5618(c_2)	-31.1247 (c_3)	

$$\Delta p_1 = \frac{b_3c_2 - b_4c_1}{c_2^2 - c_1(c_3 - b_3)}$$

$$= -0.1836$$

$$\Delta q_1 = \frac{b_4c_2 - b_3(c_3 - b_3)}{c_2^2 - c_1(c_3 - b_3)}$$

$$= -0.0505$$

Therefore

$$p_2 = p_1 + \Delta p_1 = 2.1954 - 0.1836 = 2.0118$$

$$q_2 = q_1 + \Delta q_1 = 2.0350 - 0.0505 = 1.9845$$

Therefore the required factor is $x^2 + 2.0118\,x + 1.9845$

2. Solve the equation $x^4 - 3x^3 + 20x^2 + 44x + 54 = 0$ and find the quadratic factor of the form $x^2 + px + q$ by Bairstow's method. Perform two iterations. Take $p_0 = q_0 = 2$

Solution:

	1	-3	20	44	54
$-p_0 = -2$		- 2	10	- 56	4
$-q_0 = -2$			-2	10	-56
	1	- 5	28	- 2 (b_3)	2 (b_4)
$-p_0 = -2$		- 2	14	- 80	
$-q_0 = -2$			-2	14	
	1	- 7 (c_1)	40 (c_2)	-68 (c_3)	

$$\Delta p_0 = \frac{b_3c_2 - b_4c_1}{c_2^2 - c_1(c_3 - b_3)}$$

$$= \frac{-80 + 14}{(40)^2 - 7(66)}$$

$$= -0.058$$

$$\Delta q_0 = \frac{b_4c_2 - b_3(c_3 - b_3)}{c_2^2 - c_1(c_3 - b_3)}$$

$$= \frac{80 - 2x66}{(40)^2 - 7(66)}$$

$$= -0.0457$$

Therefore

$$p_1 = p_0 + \Delta p_0 = 2 - 0.0580 = 1.942$$

$$q_1 = q_0 + \Delta q_0 = 2 - 0.0457 = 1.9543$$

Second iteration

	1	-3	20	44	54
$-p_1 = -1.942$		-1.942	9.5974	-53.6829	0.048
$-q_1 = -1.9543$			-1.9543	9.6582	-54.0229
	1	-4.942	27.6431	-0.0247 (b_3)	0.0251 (b_4)
$-p_1 = -1.942$		-1.942	13.3687	-75.8497	
$-q_1 = -1.9543$			-1.9543	13.4534	
	1	-6.884 (c_1)	39.0575 (c_2)	-64.421 (c_3)	

$$\Delta p_1 = \frac{b_3c_2 - b_4c_1}{c_2^2 - c_1(c_3 - b_3)}$$

$$= -0.007$$

$$\Delta q_1 = \frac{b_4c_2 - b_3(c_3 - b_3)}{c_2^2 - c_1(c_3 - b_3)}$$

$$= -0.0005$$

Therefore

$$p_2 = p_1 + \Delta p_1 = 1.9413$$

$$q_2 = q_1 + \Delta q_1 = 1.9538$$

Therefore the required factor is $x^2 + 1.9413\,x + 1.9538$

****

UNIT – II

Solution of Simultaneous Linear Algebraic Equations

Gauss elimination Method (Direct Method)

This is a direct method based on the elimination of the unknowns by combining equations such that the n equations in n unknowns are reduced to an equivalent upper triangular system which could be solved by back substitution.

Consider the n linear equations in n unknowns.

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots\ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots\ldots + a_{2n}x_n &= b_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \\ a_{n1}x_1 + a_{n2}x_2 + \ldots\ldots + a_{nn}x_n &= b_n \end{aligned} \qquad \ldots\ldots(1)$$

where a_{ij} and b_i are known constants and x_i's are unknowns.

The system of equations given in (1) is equivalent to

$$AX = B \qquad \ldots\ldots(2)$$

where

$$A = \begin{pmatrix} a_{11} & a_{12} & \ldots\ldots & a_{1n} \\ a_{21} & a_{22} & \ldots\ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & \ldots\ldots & a_{nn} \end{pmatrix}, \quad X = \begin{pmatrix} x_1 \\ x_2 \\ .. \\ x_n \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} b_1 \\ b_2 \\ \ldots \\ b_n \end{pmatrix}$$

The augmented coefficient matrix is given by (A,B).

$$(A, B) = \left(\begin{array}{cccc|c} a_{11} & a_{12} & \ldots\ldots & a_{1n} & b_1 \\ a_{21} & a_{22} & \ldots\ldots & a_{2n} & b_2 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & \ldots\ldots & a_{nn} & b_n \end{array}\right) \qquad \ldots\ldots(3)$$

One has to reduce the augmented matrix (A, B) given in (3) in to an upper triangular matrix. Considering a_{11} as the pivot element, multiply the first row of (3) by $(-a_{i1}/a_{11})$ and add to the i^{th} row of (A,B), where $i = 2,3, \ldots n$ so that all elements in the first column of (A, B) except a_{11} are made zero.

Now matrix (A,B) will be of the form

$$\left(\begin{array}{ccc|c} a_{11} & a_{12} \;\ldots.. & a_{1n} & b_1 \\ 0 & b_{22} \;\ldots.. & b_{2n} & c_2 \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots \\ 0 & b_{n2} \;\ldots.. & b_{nn} & c_n \end{array}\right) \qquad \ldots\ldots(4)$$

Considering b_{22} as the pivot, we have to make all elements below b_{22} in the second column of (4) as zero. This is achieved by multiplying second row of (4) by $-b_{i2}/b_{22}$ and add to the corresponding elements of the i^{th} row (i= 3, 4, … n). Now all the elements below b_{22} are reduced to zero. Now the augmented matrix in (4) has the elements as given below.

$$\left(\begin{array}{ccccc|c} a_{11} & a_{12} & a_{13} & \ldots.. & a_{1n} & b_1 \\ 0 & b_{22} & b_{23} & \ldots.. & b_{2n} & c_2 \\ 0 & 0 & c_{33} & \ldots.. & c_{3n} & d_3 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & c_{n3} & \ldots\ldots & c_{nn} & k_n \end{array}\right) \qquad \ldots\ldots(5)$$

Continuing the process, all elements below the leading diagonal elements of A are made to zero. Repeating the procedure of making the lower diagonal elements to zero for all the columns, we find that the augmented coefficient matrix (A,B) is converted into an upper triangular matrix as shown below.

$$\left(\begin{array}{cccccc|c} a_{11} & a_{12} & a_{13} & a_{14} & \ldots.. & a_{1n} & b_1 \\ 0 & b_{22} & b_{23} & b_{24} & \ldots.. & b_{2n} & c_2 \\ 0 & 0 & c_{33} & c_{34} & \ldots.. & c_{3n} & d_3 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & 0 & \ldots\ldots & \alpha_{nn} & k_n \end{array}\right) \qquad \ldots\ldots(6)$$

From (6), the given system of linear equations is equivalent to

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \;\ldots\ldots\; + a_{1n}x_n &= b_1 \\
b_{22}x_2 + b_{23}x_3 + \;\ldots\ldots\; + b_{2n}x_n &= c_2 \\
c_{33}x_3 + \;\ldots\ldots\; + c_{3n}x_n &= d_3 \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \\
\alpha_{nn}\, x_n &= k_n
\end{aligned}
$$

Going from the bottom of these equations, we solve for $x_n = k_n / \alpha_{nn}$. Using this in the penultimate equation, we get x_{n-1} and so on. By this back substitution method, we solve for

$x_n, x_{n-1}, x_{n-2}, \ldots\ldots\ldots\ldots\; x_2, x_1$

Gauss-Jordan elimination method (Direct Method)

In this method, the coefficient matrix A of the system AX= B is converted into a diagonal matrix by making all the off diagonal elements to zero by similarity transformations. Now the system (A,B) will be reduced to the form

$$
\left(\begin{array}{ccccc|c}
a_{11} & 0 & 0 & \ldots\ldots & 0 & b_1 \\
0 & b_{22} & 0 & \ldots\ldots & 0 & c_2 \\
0 & 0 & c_{33} & \ldots\ldots & 0 & d_3 \\
\ldots & \ldots & \ldots & \ldots & \ldots & . \\
0 & 0 & 0 & \ldots\ldots & \alpha_{nn} & k_n
\end{array}\right) \qquad \ldots\ldots(7)
$$

From (7) we get

$x_n = k_n / \alpha_{nn}, \ldots\ldots\; x_2 = c_2 / b_{22},\; x_1 = b_1 / a_{11}$

Problems

1. **Solve the system of equations by (i) Gauss elimination method and by (ii) Gauss-Jordan method.**

   X+2y+z=3, 2x+3y+3z=10, 3x-y+2z=13

Gauss elimination method:

The set of equations are given in matrix form as

$$\begin{pmatrix} 1 & 2 & 1 \\ 2 & 3 & 3 \\ 3 & -1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ Z \end{pmatrix} = \begin{pmatrix} 3 \\ 10 \\ 13 \end{pmatrix}$$

$$A \quad X = B$$

The augmented coefficient matrix (A,B) for the given system of equations is given as

$$(A,B) = \left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 2 & 3 & 3 & 10 \\ 3 & -1 & 3 & 13 \end{array}\right)$$

By making the transformations given by the side of the corresponding row of the matrix (A,B), we get

$$(A,B) = \left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & -1 & 1 & 4 \\ 0 & -7 & -1 & 4 \end{array}\right) \quad \begin{array}{l} \\ R_2=R_2 + (-2)R_1 \\ R_3=R_3 + (-3)R_1 \end{array}$$

Now take b_{22} = -1 as the pivot and make b_{32} as zero

$$(A,B) = \left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & -1 & 1 & 4 \\ 0 & 0 & -8 & -24 \end{array}\right) \quad R_3 = R_3 + (-7)\, R_2$$

From this, we get

$$x + 2y + z = 3$$

$$-y + z = 4$$

$$-8z = -24$$

Solving the above equations by back substitution we get

$z = 3$, $y = -1$ and $x = 2$

Gauss – Jordan method:

The upper triangular matrix (A,B) is

$$(A,B) = \left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & -1 & 1 & 4 \\ 0 & 0 & -8 & -24 \end{array}\right)$$

The off diagonal elements of (A,B) are made to zero by the following transformations given below.

$$(A,B) = \left(\begin{array}{ccc|c} 1 & 0 & 3 & 11 \\ 0 & -1 & 1 & 4 \\ 0 & 0 & -1 & -3 \end{array}\right) \qquad \begin{array}{l} R_1 = R_1 + 2\,R_2 \\ \\ R_3 = R_3 * (1/8) \end{array}$$

$$= \left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & -3 \end{array}\right) \qquad \begin{array}{l} R_1 = R_1 + 3\,R_3 \\ R_2 = R_2 + R_3 \\ \end{array}$$

Therefore we get , x = 2, y = **-**l, and z= 3. By substituting these values in the given equations, we find that the values satisfy the equations.

2. Solve the system of equations given below by Gauss elimination method.
2x + 3y – z = 5, 4x + 4y - 3z = 3 and 2x - 3y + 2z = 2

The set of equations are given in matrix form as

$$\begin{pmatrix} 2 & 3 & -1 \\ 4 & 4 & -3 \\ 2 & -3 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ Z \end{pmatrix} = \begin{pmatrix} 5 \\ 3 \\ 2 \end{pmatrix}$$

A X = B

The augmented coefficient matrix (A,B) is

$$(A,B) = \left(\begin{array}{ccc|c} 2 & 3 & -1 & 5 \\ 4 & 4 & -3 & 3 \\ 2 & -3 & 2 & 2 \end{array}\right)$$

Taking $a_{11} = 2$ as the pivot, reduce all elements below in the first column to zero

$$(A,B) = \left(\begin{array}{ccc|c} 2 & 3 & -1 & 5 \\ 0 & -2 & -1 & -7 \\ 0 & -6 & 3 & -3 \end{array}\right) \quad \begin{array}{l} \\ R_2=R_2+(-2)*R_1 \\ R_3=R_3+(-1)*R_1 \end{array}$$

Taking the element -2 in the position (2,2) as pivot, reduce the element below that to zero, we get

$$(A,B) = \left(\begin{array}{ccc|c} 2 & 3 & -1 & 5 \\ 0 & -2 & -1 & -7 \\ 0 & 0 & 6 & 18 \end{array}\right) \quad \begin{array}{l} \\ \\ R_3=R_3+(-3)*R_2 \end{array}$$

Hence $2x + 3y - z = 5$

$-2y - z = -7$

$6z = 18$

Then by back substitution we get $z = 3$, $y = 2$ and $x = 1$.

These values are substituted in the given equations and are found to satisfy them.

3. Solve the system of equations by Gauss – elimination method.

$10x - 2y + 3z = 23$

$2x + 10y - 5z = -33$

$3x - 4y + 10z = 41$

The given system of equations are written in augmented matrix form as

$$\begin{pmatrix} 10 & -2 & 3 & 23 \\ 2 & 10 & -5 & -33 \\ 3 & -4 & 10 & 41 \end{pmatrix}$$

Using the transformations given by the side, we can write

$$\begin{pmatrix} 1 & -1/5 & 3/10 & 23/10 \\ 2 & 10 & -5 & -33 \\ 3 & -4 & 10 & 41 \end{pmatrix} \quad R_1 = R_1 \div 10$$

$$\begin{pmatrix} 1 & -1/5 & 3/10 & 23/10 \\ 0 & 52/5 & -28/5 & -188/5 \\ 0 & -17/5 & 91/10 & 341/10 \end{pmatrix} \quad \begin{matrix} R_2 = R_2 - 2\,R_1, \\ R_3 = R_3 - 3R_1 \end{matrix}$$

$$\begin{pmatrix} 1 & -1/5 & 3/10 & 23/10 \\ 0 & 1 & -7/13 & -47/13 \\ 0 & -17/5 & 91/10 & 341/10 \end{pmatrix} \quad R_2 = R_2 \div 52/5$$

$$\begin{pmatrix} 1 & -1/5 & 3/10 & 23/10 \\ 0 & 1 & -7/13 & -47/13 \\ 0 & 0 & 189/26 & 567/26 \end{pmatrix} \quad R_3 = R_3 + 17/5\,R_2$$

$$\begin{pmatrix} 1 & -1/5 & 3/10 & 23/10 \\ 0 & 1 & -7/13 & -47/13 \\ 0 & 0 & 1 & 3 \end{pmatrix} \quad R_3 = R_3 \div 189/26$$

Now the coefficient matrix is upper diagonal and using back substitution we get

$Z=3$

$y - 7/13\,(z) = -47/13$

$13\,y - 7\,(3) = -47$

$13\,y = -47 + 21$

$13\,y = -26$

$Y=2$

$x - 1/5\ y + 3/10\ z = 23/10$

$10x - 2y + 3z = 23$

$10x - 2(-2) + 3(3) = 23$

$X=1$

The solution is $x=1$, $y=2$, $z=3$

4. Solve the given system of equations

$x + 3y + 3z = 16$, $x + 4y + 3z = 18$, $x + 3y + 4z = 19$

by Gauss – Jordan method.

$x + 3y + 3z = 16$

$x + 4y + 3z = 18$

$x + 3y + 4z = 19$

Write the given system of equations in augmented matrix form as

$$\begin{pmatrix} 1 & 3 & 3 & 16 \\ 1 & 4 & 3 & 18 \\ 1 & 3 & 4 & 19 \end{pmatrix}$$

Add multiples of the first row to the other rows to make all the other components in the first column equal to zero.

$$\begin{pmatrix} 1 & 3 & 3 & 16 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{pmatrix} \begin{matrix} \\ R_2 = R_2 - R_1 \\ R_3 = R_3 - R_1 \end{matrix}$$

$$\begin{pmatrix} 1 & 0 & 3 & 10 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{pmatrix} \begin{matrix} R_1 = R_1 - 3R_2, \\ \\ \\ \end{matrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{pmatrix} \quad R_1 = R_1 - 3R_3$$

The coefficient matrix finally reduces to the diagonal form and the matrix equation is given by

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

We get $x = 1,\ y = 2,\ z = 3$

5. Solve the given system of equations

$10x + y + z = 12$

$2x + 10y + z = 13$

$x + y + 5z = 7$

by Gauss – Jordan method.

The given system of equations are written in augmented matrix form as

$$\begin{pmatrix} 10 & 1 & 1 & 12 \\ 2 & 10 & 1 & 13 \\ 1 & 1 & 5 & 7 \end{pmatrix}$$

Make the element in the first row and first column as 1by

$$\begin{pmatrix} 1 & 1/10 & 1/10 & 12/10 \\ 2 & 10 & 1 & 13 \\ 1 & 1 & 5 & 7 \end{pmatrix} \quad R_1 = R_1 \div 10$$

Add multiples of the first row to the other rows and make all the other components in the first column equal to zero

$$\begin{pmatrix} 1 & 1/10 & 1/10 & 12/10 \\ 0 & 49/5 & 4/5 & 106/10 \\ 0 & 9/10 & 49/10 & 58/10 \end{pmatrix} \quad \begin{matrix} \\ R_2 = R_2 - 2R_1, \\ R_3 = R_3 - R_1 \end{matrix}$$

Make the element in the second row and second column as 1

$$\begin{pmatrix} 1 & 1/10 & 1/10 & 12/10 \\ 0 & 1 & 4/49 & 53/49 \\ 0 & 9/10 & 49/10 & 58/10 \end{pmatrix} \quad R_2 = R_2 \div 49/5$$

Add multiples of the second row to the other rows to make all the other components in the second column equal to zero.

$$\begin{pmatrix} 1 & 0 & 0.0918 & 1.0918 \\ 0 & 1 & 4/49 & 53/49 \\ 0 & 0 & 4.8265 & 4.8265 \end{pmatrix} \quad \begin{matrix} R_1 = R_1 - (1/10)R_2 \\ \\ R_3 = R_3 - (9/10)R_2, \end{matrix}$$

Make the element in the third row and third column as 1

$$\begin{pmatrix} 1 & 0 & 0.0918 & 1.0918 \\ 0 & 1 & 4/49 & 53/49 \\ 0 & 0 & 1 & 1 \end{pmatrix} \quad R_3 = R_3 \div 4.8265$$

Add multiples of the third row to the other rows to make the components in the third column equal to zero

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \quad \begin{matrix} R_1 \rightarrow R_1 - 0.0918\,R_3, \\ R_2 \rightarrow R_2 - 4/49\,R_3 \\ \\ \end{matrix}$$

The matrix finally reduces to the form given by

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Therefore $x = 1$, $y = 1$, $z = 1$

Gauss – Seidel Iterative Method

Let the given system of equations be

$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots\dots + a_{1n}x_n = C_1$

$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots\dots + a_{2n}x_n = C_2$

$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \dots\dots + a_{3n}x_n = C_3$

…………………………………………………

…………………………………………………

$a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \dots\dots + a_{nn}x_n = C_n$

The system of equations is first rewritten in the form

$x_1 = (1/a_{11})\,(C_1 - a_{12}x_2 - a_{13}x_3 - \dots\dots - a_{1n}x_n)$ …. (1)

$x_2 = (1/a_{22})\,(C_2 – a_{21}x_1 - a_{23}x_3 - \dots\dots – a_{2n}x_n)$ …. (2)

$x_3 = (1/a_{33})\,(C_3 – a_{31}x_1 - a_{32}x_2 - \dots\dots – a_{3n}x_n)$ …. (3)

……………………………………………………………..

$X_n = (1/a_{nn})\,(C_n – a_{n1}x_1 - a_{n2}x_2 - \dots\dots – a_{n,n-1}x_{n-1})$ …. (4)

First let us assume that $x_2 = x_3 = x_4 = \dots\dots = x_n = 0$ in (1) and find x_2. Let it be x_1^*. Putting x_1^* for x_1 and $x_3 = x_4 = \dots\dots = x_n = 0$ in (2) we get the value for x_2 and let it be x_2^*. Putting x_1^* for x_1 and x_2^* for x_2 and $x_3 = x_4 = \dots\dots = x_n = 0$ in (3) we get the value for x_3 and let it be x_3^*. In this way we can find the first approximate values for $x_1, x_2, \dots\dots x_n$ by using the relation

$x_1^* = (1/a_{11})\,(C_1)$

$x_2^* = (1/a_{22})\,(C_2 – a_{21}\,x_1^*)$

$x_3^* = (1/a_{33})\,(C_3 – a_{31}\,x_1^* - a_{32}\,x_2^*)$

……………………………………………………………

$X_n^* = (1/a_{nn})\ (C_n - a_{n1}x_1^* - a_{n2}x_2^* - \ldots\ldots\ldots - a_{n,n-1}x_{n-1}^*)$

Substituting these values of X* in equations (1),(2),(3),…….(4) we get

$x_1^{**} = (1/a_{11})\ (C_1 - a_{12}x_2^* - a_{13}x_3^* - \ldots\ldots\ldots - a_{1n}x_n^*)$

$x_2^{**} = (1/a_{22})\ (C_2 - a_{21}x_1^{**} - a_{23}x_3^* - \ldots\ldots\ldots - a_{2n}x_n^*)$

$x_3^{**} = (1/a_{33})\ (C_3 - a_{31}x_1^{**} - a_{32}x_2^{**} - \ldots\ldots\ldots - a_{3n}x_n^*)$

…………………………………………………………………..

$X_n^{**} = (1/a_{nn})\ (C_n - a_{n1}x_1^{**} - a_{n2}x_2^{**} - \ldots\ldots\ldots - a_{n,n-1}x_{n-1}^{**})$

Repeating this iteration procedure until two successive iterations give same value, one can get the solution for the given set of equations. This method is efficient for diagonally dominant equations.

Diagonally dominant matrix:

We say a matrix is diagonally dominant if the numerical value of the leading diagonal element (a_{ii}) in each row is greater than or equal to the sum of the numerical values of the other elements in that row.

For example the matrix $\begin{pmatrix} 5 & 1 & -1 \\ 1 & 4 & 2 \\ 1 & -2 & 5 \end{pmatrix}$ is diagonally dominant.

But the matrix $\begin{pmatrix} 5 & 1 & -1 \\ 5 & 2 & 3 \\ 1 & -2 & 5 \end{pmatrix}$ is not, since in the

Second row, the leading diagonal element 2 is less than the sum of the other two elements viz., 5 and 3 in that row.

For the Gauss – Seidel method to converge quickly, the coefficient matrix must be diagonally dominant. If it is not so, we have to rearrange the equations in such a way that the coefficient matrix is diagonally dominant and then only we can apply Gauss – Seidel method.

Problem

1. **Solve the system of equations 4x + 2y + z = 14, x + 5y – z = 10, x + y + 8z = 20 using Gauss – Seidel iteration method**.

The given system of equations is

$$4x + 2y + z = 14 \quad \ldots\ldots\ldots\ldots(1)$$

$$x + 5y - z = 10 \quad \ldots\ldots\ldots\ldots(2)$$

$$x + y + 8z = 20 \quad \ldots\ldots\ldots\ldots(3)$$

The coefficient matrix $\begin{pmatrix} 4 & 2 & 1 \\ 1 & 5 & -1 \\ 1 & 1 & 8 \end{pmatrix}$

is diagonally dominant. Hence we can apply Gauss- Seidel method.

From (1), (2) and (3) we get

$x = 1/4\ (14 - 2y - z)$ ………………....(4)

$y = 1/5\ (10 - x + z)$ ……………….(5)

$z = 1/8\ (20 - x - y)$ ……………....(6)

First Iteration

Let $y = 0$, $z = 0$ in (4) we get $x = 14/4 = 3.5$

Putting $x = 3.5$, $z = 0$ in (5) we get

$y = 1/5\ [10 - 3.5 + 0] = 1.3$

Putting $x = 3.5$, $y = 1.3$ in (6) we get

$z = 1/8\ [20 - 3.5 - 1.3] = 15.2/8 = 1.9$

Therefore in the first iteration we get $x=3.5$, $y=1.3$ and $z=1.9$

Second Iteration

Putting $y = 1.3$, $z = 1.9$ in (4) we get

$x = 1/4\ [14 - 2\ (1.3) - 1.9] = 2.375$

Putting $x = 2.375$, $z = 1.9$ in (5) we get

$y = 1/5\ [10 - 2.375 + 1.9] = 1.905$

Putting $x = 2.375$, $y = 1.905$ in (6) we get

$z = 1/8\ [20 - 2.375 - 1.905] = 1.965$

In the second iteration we get $x=2.375$, $y=1.905$ and $z=1.965$

Third Iteration

Putting y = 1.905, z = 1.965 in (4) we get

x = 1/4 [14 – 2 (1.905) – 1.965] = 2.056

Putting x = 2.056, z = 1.965 in (5) we get

y = 1/5 [10 – 2.056 + 1.965] = 1.982

Putting x = 2.056, y = 1.982 in (6) we get

z = 1/8 [20 – 2.0565 – 1.982] = 1.995

In the third iteration we get x=2.056, y= 1.982, z=1.995

Fourth Iteration

Putting y = 1.982, z = 1.995 in (4) we get

x = 1/4 [14 – 2 (1.982) – 1.995] = 2.010

Putting x = 2.010 z = 1.995 in (5) we get

y = 1/5 [10 – 2.010 + 1.995] = 1.997

Putting x = 2.010, y = 1.997 in (6) we get

z = 1/8 [20 – 2.010 – 1.997] = 1.999

In the fourth iteration we get x=2.01, y=1.997, z=1.999

Fifth Iteration

Putting y = 1.997, z = 1.999 in (4) we get

x = 1/4 [14 – 2 (1.997) – 1.999] = 2.001

Putting x = 2.001 z = 1.999 in (5) we get

y = 1/5 [10 – 2.001 + 1.999] = 1.999

Putting x = 2.010, y = 1.999 in (6) we get

z = 1/8 [20 – 2.001 – 1.999] = 2

In the fifth iteration we get x=2, y=2 and z=2 and the values satisfy the equations.

2. Solve the system of equations x + y + 54 z = 110, 27 x + 6 y – z = 85,

6 x + 15 y + 2 z = 72 using Gauss – Seidel iteration method.

The given system is

$x + y + 54z = 110,$

$27x + 6y - z = 85,$

$6x + 15y + 2z = 72$

Interchanging the equations

$27x + 6y - z = 85,$ …. (1)

$6x + 15y + 2z = 72$ …. (2)

$x + y + 54z = 110,$ …. (3)

we get a diagonally dominant system of equations.

From (1), (2) and (3) we get

$x = 1/27\ (85 - 6y + z)$ ………………...(4)

$y = 1/15\ (2 - 6x - 2z)$ ……………….(5)

$z = 1/54\ (110 - x - y)$ ………………...(6)

First Iteration

Putting y = 0, z = 0 in (4) we get x = 85/27 = 3.148

Putting x = 3.148, z = 0 in (5) we get

$y = 1/15\ [72 - 3.148 - 2(0)] = 3.5408$

Putting x = 3.148, y = 3.5408 in (6) we get

$z = 1/54\ [110 - 3.148 - 3.5408] = 1.913$

In the first iteration we get x=3.148, y=3.5408, z=1.913

Second Iteration

Putting y = 3.5408, z = 1.913 in (4) we get

$x = 1/27\ [85 - 6(3.5408) - 1.913] = 2.4322$

Putting x = 2.4322, z = 1.913 in (5) we get y = 1/15 [72 – 2.4322 – 2 (1.913)] = 3.572

Putting x = 2.4322, y = 3.572 in (6) we get

z = 1/54 [110 – 2.4322 – 3.572] = 1.92585

In the second iteration we get x=2.4322, y=3.572, z=1.92585

Third Iteration

Putting y = 3.572, z = 1.92585 in (4) we get

x = 1/27 [85 – 6 (3.572) – 1.92585] = 2.42569

Putting x = 2.42569, z = 1.92585 in (5) we get

y = 1/15 [72 – 2.42569 – 2 (1.92585)] = 3.5729

Putting x = 2.42569, y = 3.5729 in (6) we get

z = 1/54 [110 – 2.42569 – 3.5729] = 1.92595

In the third iteration we get x=2.42569, y=3.5729, z=1.92595

Fourth Iteration

Putting y = 3.5729, z = 1.92595 in (4) we get

x = 1/27 [85 – 6 (3.5729) – 1.92595] = 2.42550

Putting x = 2.42550, z = 1.92595 in (5) we get

y = 1/15 [72 – 2.42550 – 2 (1.92595)] = 3.5730

Putting x = 2.42550, y = 3.573 in (6) we get

z = 1/54 [110 – 2.42550 – 3.573] = 1.92595

In the fourth iteration we get x=2.425, y=3.573, z= 1.92595

Since the values of successive iterations are same, the answer is

x=2.425, y=3.573, z= 1.92595

Jacobi Method

If A is a real symmetric matrix and R is an orthogonal matrix then the transformation given by

$$B \quad = \quad R^{1}AR \; = \quad R^{-1}AR \qquad (\text{since } R^{1} = R^{-1})$$

transforms A into B and this transformation preserves the symmetry and the eigen values of B. That is, B is also symmetric and the eigen values of B are also those of A. If X is the eigen vector corresponding to the eigen value λ of A_1 then the eigen vector of B corresponding to λ is $R^{-1}X$. Further if B happens to be in diagonal matrix, the eigen values of B and hence of A are the diagonal elements of B.

Rotation Matrix

If P(x, y) is any point in the xy plane and if OP is rotated (O is the origin) in the clockwise direction through an angle θ, then the new points of P (x', y') is given by

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \text{Cos}\theta & -\text{Sin}\theta \\ \text{Sin}\theta & -\text{Cos}\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

i.e.

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = R \begin{bmatrix} x \\ y \end{bmatrix}$$

where $R = \begin{bmatrix} \text{Cos}\theta & -\text{Sin}\theta \\ \text{Sin}\theta & -\text{Cos}\theta \end{bmatrix}$

Hence R is called a Rotation Matrix in the xy plane. Here R is also a orthogonal matrix since RR' = I

Let S_1 be a n x n matrix in which the diagonal elements are 1and all other off diagonal elements are zero except

$$a_{ii} = \text{Cos}\theta, \quad a_{jj} = \text{Cos}\theta, \quad a_{ij} = -\text{Sin}\theta, \quad a_{ji} = \text{Sin}\theta$$

Now S_1 is orthogonal.

Perform $S_1^{-1}AS_1 = B_1$ and this operation annihilates all b_{ij} to zero in B_1.

In the next step, take the largest off-diagonal element b_{kl} in B_1 and annihilate to get

$$B_2 = S_2^{-1}B_1S_2$$

Performing series of such rotation by S_1, S_2, S_3, ……. After K operations, we get

$$B_k = S_k^{-1} S_{k-1}^{-1} \ldots\ldots S_1^{-1} AS_1S_2 \ldots\ldots S_k$$

$$= (S_1S_2 \ldots. S_k)^{-1} A (S_1S_2 \ldots. S_k) = S^{-1}AS$$

Where $S = S_1S_2 \ldots\ldots S_k$.

If B_k is diagonal matrix, we get immediately eigen values of B_1 and hence of A.

Since an element annihilated by one plane rotation may not necessarily remain zero during subsequent transformations, the operations may continue so that $k \rightarrow \infty$. As $k \rightarrow \infty$, B_k will approach a diagonal matrix. The minimum number of rotations required to bring matrix A of order n into a diagonal form is n (n-1)/2.

Consider a third order matrix A. we will apply the above procedure to get the eigen values for eigen vectors of A. Choose the largest off diagonal element of A and let it be a_{ij}. The matrix will be reduced into a diagonal one if the rotation angle is given by $\tan 2\theta = 2a_{ij} / (a_{ii} - a_{jj})$.

That is $\theta = \frac{1}{2} \tan^{-1} (2a_{ij} / (a_{ii} - a_{jj}))$.

As an example, if a_{13} is the largest off diagonal element of matrix A, then

$$\theta = \frac{1}{2} \tan^{-1} (2a_{13} / (a_{11} - a_{33}))$$

and the rotation matrix S_1 is

$$\begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix}$$

And S_1^{-1} is given by

$$\begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}$$

So that $B_1 = S_1^{-1} A S_1$

Problems:

1. Find the eigen values and eigen vectors of the matrix by Jacobi's method.

$$A = \begin{pmatrix} 1 & \sqrt{2} & 2 \\ \sqrt{2} & 3 & \sqrt{2} \\ 2 & \sqrt{2} & 1 \end{pmatrix}$$

Solution:

Here the largest off-diagonal element is $a_{13} = a_{31} = 2$ and $a_{11} = a_{33} = 1$

Hence take the rotation matrix

$$S_1 = \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix}$$

Now $\theta = \frac{1}{2} \tan^{-1} (2a_{13} / (a_{11} - a_{33})) = \frac{1}{2} \tan^{-1} (4 / 0) = \frac{1}{2} \tan^{-1} \infty = (\frac{1}{2})(\pi/2)$

ie. $\theta = \pi/4$,

$$S_1 = \begin{pmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{pmatrix}$$

$$B_1 = S_1^{-1} A S_1$$

$$= \begin{pmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 1 & \sqrt{2} & 2 \\ \sqrt{2} & 3 & \sqrt{2} \\ 2 & \sqrt{2} & 1 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{pmatrix}$$

$$= 1/2 \begin{pmatrix} 1 & 0 & 1 \\ 0 & \sqrt{2} & 0 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \sqrt{2} & 2 \\ \sqrt{2} & 3 & \sqrt{2} \\ 2 & \sqrt{2} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 \\ 0 & \sqrt{2} & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 1 & 0 & 1 \\ 0 & \sqrt{2} & 0 \\ -1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 3 & 2 & 1 \\ 2\sqrt{2} & 3\sqrt{2} & 0 \\ 3 & 2 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Again annihilate the largest off-diagonal element $a_{12} = a_{21} = 2$ in B_1

Also $a_{11} = 3$ $a_{22} = 3$

Take

$$S_2 = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \text{Sin}\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Now $\theta = \frac{1}{2} \tan^{-1} (2a_{12} / (a_{11} - a_{22})) = \frac{1}{2} \tan^{-1} (4 / 0) = \frac{1}{2} \tan^{-1} \infty = (\frac{1}{2})(\pi/2)$

ie. $\theta = \pi/4$,

$$S_2 = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad S_2^{-1} = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$B_2 = S_2^{-1} B_1 S_2$$

$$= \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 3 & 2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \quad 1/2 \begin{pmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix} \begin{pmatrix} 3 & 2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix}$$

$$= \quad 1/2 \begin{pmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix} \begin{pmatrix} 5 & -1 & 0 \\ 5 & 1 & 0 \\ 0 & 0 & -\sqrt{2} \end{pmatrix}$$

$$= \begin{pmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

After 2 rotations, A is reduced to the diagonal matrix B_2. Hence the eigen values of A are 5, 1, -1

$$S = S_1 S_2$$

$$= \begin{pmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \quad 1/2 \begin{pmatrix} 1 & 0 & -1 \\ 0 & \sqrt{2} & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix}$$

$$= \begin{pmatrix} 1/2 & -1/2 & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{1} & 0 \\ ½ & -1/2 & 1/\sqrt{2} \end{pmatrix}$$

Hence the corresponding vectors are $\begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}$, $\begin{pmatrix} -1 \\ \sqrt{2} \\ -1 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

2. Find the eigen values and eigen vectors of the matrix by Jacobi's method.

$$A = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix}$$

Solution:

Here the largest off-diagonal element is $a_{13} = 1$

Let us annihilae this element

Take $S_1 = \begin{pmatrix} \text{Cos}\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \text{Sin}\theta & 0 & \cos\theta \end{pmatrix}$

$\tan 2\theta = 2a_{13}/(a_{11} - a_{33}) = 2 / (2 - 2) = \infty$

$\theta = \pi/4,$

$$S_1 = \begin{pmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{pmatrix}$$

$B_1 = S_1^{-1} AS_1$

$$= \begin{pmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{pmatrix}$$

$$= 1/2 \begin{pmatrix} 1 & 0 & 1 \\ 0 & \sqrt{2} & 0 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 \\ 0 & \sqrt{2} & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

$$= 1/2 \begin{pmatrix} 1 & 0 & 1 \\ 0 & \sqrt{2} & 0 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & 0 & -1 \\ 0 & 2\sqrt{2} & 0 \\ 3 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Since B_1 is a diagonal matrix the eigen values of A are 3, 2, 1

The corresponding eigen vectors of A are the columns of S_1, namely

$$\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

3. Find the eigen values and eigen vectors of the given matrix by Jacobi's method.

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{pmatrix}$$

Solution:

The largest off diagonal element is $a_{23} = 1$ and this is to be annihilated.

Therefore, the rotation matrix S_1 will have $a_{22} = a_{33} = \cos\theta$, $a_{23} = -\sin\theta$, $a_{32} = \sin\theta$

$a_{11} = 1$, $a_{12} = a_{21} = 0$

ie.

$$S_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$$

$$\tan 2\theta = 2a_{23} / (a_{22} - a_{33}) = -2 / 0 = \infty$$

$$\theta = \pi/4,$$

$$S_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$$

$$B_1 = S_1^{-1} A S_1$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$$

$$= 1/2 \begin{pmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{pmatrix} \begin{pmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{pmatrix}$$

$$= 1/2 \begin{pmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{2} & 0 & 0 \\ 0 & 2 & -4 \\ 0 & 2 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

Therefore the eigen values of A are 1, 2, 4

The corresponding eigen vectors of A are the columns of S_1, namely

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \text{ and } \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

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UNIT III - INTERPOLATION

INTERPOLATION WITH UNEQUAL INTERVALS

Lagrange's Interpolation Formula for unequal intervals

Let $f(x_0)$, $f(x_1)$, …… $f(x_n)$ be the values of the function $y = f(x)$ corresponding to the arguments x_0. x_1, x_2, …….. x_n. The arguments are not equally spaced. Let $f(x)$ be a polynomial in x of degree n. $f(x)$ can be written as

$$f(x) = a_0 (x - x_1)(x - x_2) \ldots\ldots\ldots (x - x_n)$$
$$+ a_1 (x - x_0)(x - x_2) \ldots. (x - x_n) + \ldots\ldots$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$+ a_n (x - x_0)(x - x_1) \ldots.. (x - x_{n-1}) \quad \ldots. (1)$$

Where $a_0, a_1, \ldots.. a_n$ are constants. Their values can be obtained by

replacing $x = x_0$ in (1) we get

$$f(x_0) = a_0 (x_0 - x_1)(x_0 - x_2) \ldots\ldots\ldots (x_0 - x_n)$$

$$\text{i.e.} \quad a_0 = \left[\frac{f(x_0)}{(x_0 - x_1)(x_0 - x_2) \ldots\ldots\ldots (x_0 - x_n)} \right] \quad \ldots. (2)$$

Putting $x = x_1$ in (1) we get

$$f(x_1) = a_1 (x_1 - x_0)(x_1 - x_2) \ldots\ldots\ldots (x_1 - x_n)$$

$$\text{i.e.} \quad a_1 = \left[\frac{f(x_1)}{(x_1 - x_0)(x_1 - x_2) \ldots\ldots\ldots (x_1 - x_n)} \right] \quad \ldots. (3)$$

Similarly

$$. \quad a_2 = \left[\frac{f(x_2)}{(x_2 - x_0)(x_2 - x_1) \ldots\ldots\ldots (x_2 - x_n)} \right] \quad \ldots. (4)$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$a_n = \left[\frac{f(x_n)}{(x_n - x_0)(x_n - x_1) \ldots\ldots\ldots (x_0 - x_{n-1})} \right] \quad \ldots. (5)$$

Substituting (2), (3), (4), (5) in (1) we get

$$f(x) = \left[\frac{(x-x_1)(x-x_2)\ldots\ldots.(x-x_n)}{(x_0-x_1)(x_0-x_2)\ldots\ldots\ldots(x_0-x_n)}\right] f(x_0)$$

$$+ \left[\frac{(x-x_0)(x-x_2)\ldots\ldots.(x-x_n)}{(x_1-x_0)(x_1-x_2)\ldots\ldots\ldots(x_1-x_n)}\right] f(x_1)$$

$$+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..$$

$$+ \left[\frac{(x-x_0)(x-x_1)\ldots\ldots.(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\ldots(x_n-x_{n-1})}\right] f(x_n)$$

If we denote $f(x_0)$, $f(x_1)$, ….. $f(x_n)$ by y_0, y_1, …… y_n, we get

$$f(x) = \left[\frac{(x-x_1)(x-x_2)\ldots\ldots.(x-x_n)}{(x_0-x_1)(x_0-x_2)\ldots\ldots\ldots(x_0-x_n)}\right] y_0$$

$$+ \left[\frac{(x-x_0)(x-x_2)\ldots\ldots.(x-x_n)}{(x_1-x_0)(x_1-x_2)\ldots\ldots\ldots(x_1-x_n)}\right] y_1$$

$$+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..$$

$$+ \left[\frac{(x-x_0)(x-x_1)\ldots\ldots.(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\ldots\ldots\ldots(x_n-x_{n-1})}\right] y_n \qquad \ldots\ldots.(6)$$

and this is known as **Lagrange's Interpolation formula**.

Problem:

1. Using Lagrange's interpolation formula calculate y(3) from the data given below.

x	0	1	2	4	5	6
y (x)	1	14	15	5	6	19

Here $x_0 = 0$ $x_1 = 1$ $x_2 = 2$ $x_3 = 4$ $x_4 = 5$ $x_5 = 6$

$Y_0 = 1$ $y_1 = 14$ $y_2 = 15$ $y_3 = 5$ $y_4 = 9$ $y_5 = 19$

Lagrange's interpolation formula is

$$y(x) = \left[\frac{(x - x_1)(x - x_2)(x - x_3)(x - x_4)(x - x_5)}{(x_0 - x_1)(x_0 - x_2)(x_0 - x_3)(x_0 - x_4)(x_0 - x_5)}\right] y_0$$

$$+ \left[\frac{(x - x_0)(x - x_2)(x - x_3)(x - x_4)(x - x_5)}{(x_1 - x_0)(x_1 - x_2)(x_1 - x_3)(x_1 - x_4)(x_1 - x_5)}\right] y_1$$

$$+ \left[\frac{(x - x_0)(x - x_1)(x - x_3)(x - x_4)(x - x_5)}{(x_2 - x_0)(x_2 - x_1)(x_2 - x_3)(x_2 - x_4)(x_2 - x_5)}\right] y_2$$

$$+ \left[\frac{(x - x_0)(x - x_1)(x - x_2)(x - x_4)(x - x_5)}{(x_3 - x_0)(x_3 - x_1)(x_3 - x_2)(x_3 - x_4)(x_3 - x_5)}\right] y_3$$

$$+ \left[\frac{(x - x_0)(x - x_1)(x - x_2)(x - x_3)(x - x_5)}{(x_4 - x_0)(x_4 - x_1)(x_4 - x_2)(x_4 - x_3)(x_4 - x_5)}\right] y_4$$

$$+ \left[\frac{(x - x_0)(x - x_1)(x - x_2)(x - x_3)(x - x_4)}{(x_5 - x_0)(x_5 - x_1)(x_5 - x_2)(x_5 - x_3)(x_5 - x_4)}\right] y_5$$

Substituting the values, we can write

$$y(x) = \left[\frac{(x-1)(x-2)(x-4)(x-5)(x-6)}{(0-1)(0-2)(0-4)(0-5)(0-6)}\right] X\ 1$$

$$+ \left[\frac{(x-0)(x-2)(x-4)(x-5)(x-6)}{(1-0)(1-2)(1-4)(1-5)(1-6)}\right] X14$$

$$+ \left[\frac{(x-0)(x-1)(x-4)(x-5)(x-6)}{(2-0)(2-1)(2-4)(2-5)(2-6)}\right] X15$$

$$+ \left[\frac{(x-0)(x-1)(x-2)(x-5)(x-6)}{(4-0)(4-1)(4-2)(4-5)(4-6)}\right] X5$$

$$+ \left[\frac{(x-0)(x-1)(x-2)(x-4)(x-6)}{(5-0)(5-1)(5-2)(5-4)(5-6)}\right] X6$$

$$+ \left[\frac{(x-0)(x-1)(x-2)(x-4)(x-5)}{(6-0)(6-1)(6-2)(6-4)(6-5)}\right] X19$$

Substituting $x = 3$, we get $y(3) = 10$.

2. using Lagrange's interpolation formula for unequal intervals find the value of y when x = 10 using the values of x and y given below.

x	5	6	9	11
y	12	13	14	16

Given: $x_0 = 5$, $x_1 = 6$, $x_2 = 9$, $x_3 = 11$, $y_0 = 12$ $y_1 = 12$ $y_2 = 14$ $y_3 = 16$

Lagrange's interpolation formula is

$$y(x) = \left[\frac{(x - x_1)(x - x_2)(x - x_3)}{(x_0 - x_1)(x_0 - x_2)(x_0 - x_3)}\right] y_0$$

$$+ \left[\frac{(x - x_0)(x - x_2)(x - x_3)}{(x_1 - x_0)(x_1 - x_2)(x_1 - x_3)}\right] y_1$$

$$+ \left[\frac{(x - x_0)(x - x_1)(x - x_3)}{(x_2 - x_0)(x_2 - x_1)(x_2 - x_3)}\right] y_2$$

$$+ \left[\frac{(x - x_0)(x - x_1)(x - x_2)}{(x_3 - x_0)(x_3 - x_1)(x_3 - x_2)}\right] y_3$$

Substituting the values with x=10, we get

$$y(10) = \left[\frac{(10 - 6)(10 - 9)(10 - 11)}{(5 - 6)(5 - 9)(5 - 11)}\right] \times (12)$$

$$+ \left[\frac{(10 - 5)(10 - 9)(10 - 11)}{(6 - 5)(6 - 9)(6 - 11)}\right] \times (13)$$

$$+ \left[\frac{(10 - 5)(10 - 6)(10 - 11)}{(9 - 5)(9 - 6)(9 - 11)}\right] \times (14)$$

$$+ \left[\frac{(10 - 5)(10 - 6)(10 - 9)}{(11 - 5)(11 - 6)(11 - 9)}\right] \times (16)$$

$= \quad 2 \; - \; 4.33 \; + \; 11.66 \; + \; 5.3$

$Y(10) = 14.63$

3. Use Lagrange's interpolation formula to find the form of the function for the Data given below.

x	3	2	1	-1
f (x)	3	12	15	-21

Solution:

Here $x_0 = 3, x_1 = 2, x_2 = 1, x_3 = -1, y_0 = 3, y_1 = 12, y_2 = 15, y_3 = -21$

$$f(x) = \left[\frac{(x-2)(x-1)(x+1)}{(3-2)(3-1)(3+1)}\right] \times 3$$

$$+ \left[\frac{(x-3)(x-1)(x+1)}{(2-3)(2-1)(2+1)}\right] \times 12$$

$$+ \left[\frac{(x-3)(x-1)(x+1)}{(1-3)(1-2)(1+1)}\right] \times 15$$

$$+ \left[\frac{(x-3)(x-1)(x+1)}{(-1-3)(-1-3)(-1-1)}\right] \times (-21)$$

Simplifying, we get $f(x) = x^3 - 9x^2 + 17x + 6.$

f(x) is the required form of function for the given data.

Newton's Forward Interpolation Formula

We know that

$$\Delta y_0 = y_1 - y_0 \quad \text{i.e.} \quad y_1 = y_0 + \Delta y_0 = (1 + \Delta)\, y_0$$

$$\Delta y_1 = y_2 - y_1 \quad \text{i.e.} \quad y_2 = y_1 + \Delta y_1 = (1 + \Delta)\, y_1 = (1 + \Delta)^2\, y_0$$

$$\Delta y_2 = y_3 - y_1 \quad \text{i.e.} \quad y_3 = y_2 + \Delta y_2 = (1 + \Delta)\, y_2 = (1 + \Delta)^3\, y_0$$

In general $y_n = (1 + \Delta)^n\, y_0$

Expanding $(1 + \Delta)^n$ by using Binomial theorem we have

$$y_n = \left[1 + n\Delta + \frac{n(n-1)}{2!}\Delta^2 + \frac{n(n-1)(n-2)}{3!}\Delta^3 + \dots \dots \right] y_0$$

$$y_n = \left[y_0 + n\Delta y_0 + \frac{n(n-1)}{2!}\Delta^2 y_0 + \frac{n(n-1)(n-2)}{3!}\Delta^3 y_0 + \dots \dots \right]$$

This result is known as Gregory-Newton forward interpolation formula(or) Newton's formula for equal intervals.

Problem:

1. **The following table gives the population of a town taken during six censuses. Estimate the increase in the population during the period 1946 to 1948**.

Year	1911	1921	1931	1941	1951	1961
Population (in thousands)	12	13	20	27	39	52

The difference table is givenbelow.

x	y = f(x)	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
1911 (x_o)	12 (y_0)					
		1 (Δy_0)				
1921	13		6 ($\Delta^2 y_0$)			
		7		-6 ($\Delta^3 y_0$)		
1931	20		0		11 ($\Delta^4 y_0$)	
		7		5		-20 ($\Delta^5 y_0$)
1941	27		5		-9	
		12		-4		
1951	39		1			
		13				
1961	52					

Here $x_0 = 1911$ $h = 10$ $y_0 = 12$

Newton's forward interpolation formula is

$$y(x_0 + nh) = y_0 + n\Delta y_0 + \frac{n(n-1)}{2!}\Delta^2 y_0 + \frac{n(n-1)(n-2)}{3!}\Delta^3 y_0 + \ldots\ldots\ldots$$

The population in the year 1946 ie y(1946) is to be calculated.

Here $x_0 + nh = 1946$

i.e., $1911 + n \times 10 = 1946$ i.e., $n = 3.5$

Therefore

$$y(1946) = 12 + (3.5)(1) + \frac{(3.5)(3.5-1)}{2} \times 6$$

$$+ \frac{(3.5)(3.5-1)(3.5-2)}{6} \times (-6)$$

$$+ \frac{(3.5)(3.5-1)(3.5-2)(3.5-3)}{24} \times (11)$$

$$+ \frac{(3.5)(3.5-1)(3.5-2)(3.5-3)(3.5-4)}{120} \times (-20)$$

$$= 12 + 3.5 + 26.25 - 13.125 + 3.0078 + 0.5469$$

$$= 32.18 \text{ Thousands}$$

The population in the year 1948 ie y (1948) is calculated as below.

Here $x_0 + nh = 1948$

i.e. $1911 + n.10 = 1948$, i.e., $n = 3.7$

Therefore $y(1946) = 12 + (3.7)(1) + \frac{(3.7)(3.7-1)}{2} \times 6$

$$+ \frac{(3.7)(3.7-1)(3.7-2)}{6} \times (-6)$$

$$+ \frac{(3.7)(3.7-1)(3.7-2)(3.7-3)}{24} \times (11)$$

$$+ \frac{(3.7)(3.7-1)(3.7-2)(3.7-3)(3.7-4)}{120} \times (-20)$$

$$= 12 + 3.7 + 29.97 - 16.983 + 5.4487 + 0.5944$$

$$= 34.73 \text{ thousands}$$

Increase in the population during the period 1946 to 1948 is

= Population in 1948 - Population in 1946

= 34.73 - 32.18 = **2.55** thousands.

2. From the following data, find θ at x = 43.

x	40	50	60	70	80	90
θ	184	204	226	250	276	304

Since the vale to be interpolated is at the beginning of the table, we can use Newton's forward interpolation formula.

The difference table is given below.

x	y = f(x)	Δ y	Δ^2 y	Δ^3 y
40 (=x_0)	184 (=y_0)			
		20 (=Δy_0)		
50	204		2(=$\Delta^2 y_0$)	
		22		0 (=$\Delta^3 y_0$)
60	226		2	
		24		0
70	250		2	
		26		0
80	276		2	
		28		
90	304			

Here $x_0 = 40$ $h = 10$ $y_0 = 184$

By Newton's formula we have

$$y(x_0 + nh) = y_0 + n\Delta y_0 + \frac{n(n-1)}{2!}\Delta^2 y_0 + \frac{n(n-1)(n-2)}{3!}\Delta^3 y_0 + \ldots\ldots\ldots$$

We have to find the value of y at x=43 ie. y(43)

i.e., $x_0 + nh = 43$

i.e., $40 + n.10 = 43$ i.e., $n = (43 - 40)/10 = 0.3$

Therefore $y(43) = 184 + (0.3)(20) + \frac{(0.3)(0.3 - 1)}{2} \times (2)$

$= 184 + 6 - 0.21 =$ **189.79**

Newton – Gregory Formula for Backward Interpolation

The backward difference formula for any function f (x) is given as

$\nabla f(x) = f(x) - f(x - h)$, h being interval of data x. If f (a), f(a+h), f(a+2h) ……….. f(a+nh) are the (n+1) values for the function f (x) corresponding to an independent values of x at equal intervals x = a, a+h, a+2h, …. (a+nh). We can write f(x) in a polynomial of the form

$$f(x) = a_0 + a_1\{x - (a+nh)\} + a_2\{x - (a+nh)\}\{x - a + (n-1)h\}$$

$$+ \ldots\ldots + a_n\{x - (a+nh)\}\{x + a + (n-1)h\} \ldots \{x - (a+h)\} \quad \ldots(1)$$

Where the constant a_0, a_1, a_2, …… a_n, are to be determined.

Putting x = a + nh, $a + \overline{n-1}\,h$, ….. a in succession in (1) we get

$$f(a+nh) = a_0 \quad \text{ie. } a_0 = f(a+nh)$$

$$f(a+(n-1)h) = a_0 + a_1(-h)$$

$$\text{ie. } a_1 = \frac{f(a+nh) - f(a+(n-1)h)}{h} = \frac{\nabla f(a+nh)}{h}$$

Similarly, $a_2 = \dfrac{\nabla^2 f(a+nh)}{2!\,h^2}$ $a_n = \dfrac{\nabla^n f(a+nh)}{n!\,h^n}$

Substituting these values of a_0, a_1, a_2, ……. a_n in (1) we get

$$f(x) = f(a+nh) + \{x - (a-nh)\}\frac{\nabla f(a+nh)}{h} +$$

$$+\{x-(a+nh)\}\{x-(a+\overline{n-1}\,h)\} * \frac{\nabla^2 f(a+nh)}{2!\,h^2}$$

$$+\{x-(a+nh)\}\{x-(a+\overline{n-1}\,h)\} \ldots \{x-(a+h)\} * \frac{\nabla^n f(a+nh)}{n!\,h^n} \quad \ldots (2)$$

This is Newton – Gregory formula for backward interpolation.

Putting $u = x - (a + nh) / h$ or $x = a + nh + hu$ in (2) we get

$$f(x) = f[a + h(u+n)]$$

$$= f(a + nh) + u \nabla f(a + nh) + \frac{u(u+1)}{2!} \nabla^2 f(a + nh) + \ldots\ldots$$

$$+ \frac{u(u+1)\ldots\ldots(u+n-1) \nabla^n f(a + nh)}{n!}$$

This is the usual form of Newton-Gregory interpolation formula.

Problems:

1. Using Newton's backward interpolation method, fit a polynomial of degree three for the given data.

x	3	4	5	6
y	6	24	60	120

Solution

The Newton-Gregory backward interpolation formula is

$$y(x_0 + nh) = y_0 + n\nabla y_0 + \frac{n(n-1)}{2!} \nabla^2 y_0 + \frac{n(n-1)(n-2)}{3!} \nabla^3 y_0 + \ldots\ldots\ldots$$

Here $x_0 + nh = x$ $x_0 = 6$, $n=?$, $h=1$

$n = x-6$

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$
3	6	18		
4	24	36	18	$6=\nabla^3 y_0$
5	60	$60=\nabla y_0$	$24= \nabla^2 y_0$	
$6= x_0$	$120=y_0$			

$Y(x) = 120 + (x-6)\,60 + (1/2)(x-6)(x-5)\,24 + (1/6)(x-6)(x-5)(x-4)\,6$

Therefore $Y(x) = x^3 - 3x^2 + 2x$

2. Find the value of y(63) from the data given below.

x	45	50	55	60	65
y	114.84	96.16	83.32	74.48	68.48

Solution

The Newton-Gregory backward interpolation formula is

$$y(x_0 + nh) = y_0 + n\nabla y_0 + \frac{n(n-1)}{2!}\nabla^2 y_0 + \frac{n(n-1)(n-2)}{3!}\nabla^3 y_0 + \ldots\ldots\ldots$$

Here, $x_0 + nh = x$ $\quad x_0 = 65$, $h=5$, $x= 63$, $n=?$

$n = (63 - 65)/5 = -2/5$

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
45	114.84	-18.68			
50	96.16	-12.84	5.84	-1.84	
55	83.32	-8.84	4	$-1.16=\nabla^3 y_0$	$0.68=\nabla^4 y$
60	74.48	$-6=\nabla y_0$	$2.84=\nabla^2 y_0$		
$65=x_0$	$68.48=y_0$				

$$Y(63) = 68.48 + (-\tfrac{2}{5})(-6) + (\tfrac{1}{2!})(-\tfrac{2}{5})(-\tfrac{2}{5}+1)(2.84) + (\tfrac{1}{3!})(-\tfrac{2}{5})(-\tfrac{2}{5}+1)(-\tfrac{2}{5}+2)(-1.16)$$

$$+ (\tfrac{1}{4!})(-\tfrac{2}{5})(-\tfrac{2}{5}+1)(-\tfrac{2}{5}+2)(-\tfrac{2}{5}+3)\ (0.68)$$

Y(63) = 68.48+2.4 – 0.3408 + 0.07424 – 0.028288

Y(63) = 70.585

3. Using Newton's backward interpolation formula evaluate y(9.5) from the data given below.

x	6	7	8	9	10
y	46	66	81	93	101

Solution

$$y(x_0 + nh) = y_0 + n\nabla y_0 + \frac{n(n-1)}{2!}\nabla^2 y_0 + \frac{n(n-1)(n-2)}{3!}\nabla^3 y_0 + \ldots\ldots\ldots$$

Here, $x_0 + nh = x$ $x_0 = 10$, h=10, x= 9.5, n=?

n = (9.5 – 10)/10 = - 0.5

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
6	46	20			
7	66	15	-5	2	
8	81	12	-3	$-1=\nabla^3 y_0$	$-3=\nabla^4 y$
9	93	$8=\nabla y_0$	$-4=\nabla^2 y_0$		
$10=x_0$	$101=y_0$				

$$Y(9.6) = 101 + (-0.5)(8) + \left(\frac{1}{2!}\right)(-0.5)(-0.5+1)(-4) + \left(\frac{1}{3!}\right)(-0.5)(-0.5+1)(-0.5+2)(-1)$$

$$+ \left(\frac{1}{4!}\right)(-0.5)(-0.5+1)(-0.5+2)(-0.5+3)(-3)$$

Y(9.6) = 101-4 + 0.5 + 0.0625+0.1172

Y(9.6) = 97.68

The Method of Least Squares:

Given n-paired observations (x_1, y_1), (x_2, y_2), ……. (x_n, y_n) of two variables x and y and we want to determine a function f (x) such that

$f(x_i) = y_i$, i = 1, 2, 3, ……. n

we have to fit a straight line of the form y = a + bx and choose the parameters a and b such that the sum of squares of deviations of the fitted value with the given data is least or minimum. This method of fitting a curve is called **least squares fitting method.** The sum of squares of deviations of the fitted value with the given value is given by

s = $\sum_{1}^{n}[f(x_i) - y_i]^2$ is a function of a_1, a_2, a_3,…. a_n

According to the principle of least squares a's may be determined by the requirement that $\sum_{i} s$ is least i.e.

$\partial s / \partial a_1 = 0$, $\partial s / \partial a_2 = 0$, …… $\partial s / \partial a_i = 0$,

A set of these equations is called the normal equations. The unknowns in y = f (x) are determined using these normal equations. For the linear equation y=a+bx

the residuals are given by $v_i = a + bx_i - y_i$, and the sum of residues is given by

s = $\sum_{1}^{n}[a + bx_i - y_i]^2$ Differentiating with respect to a and b, we get

$$\partial s / \partial a = 2. \sum_{1}^{n}[a + bx_i - y_i] = 0$$

and $$\partial s / \partial b = 2. \sum_{1}^{n} x_i[a + bx_i - y_i] = 0$$

collecting the coefficients of a and b in the above equations, we get

$$\mathbf{na + b} \sum_{i=1}^{n} x_i \;\; \mathbf{=} \;\; \sum_{i=1}^{n} y_i \qquad \text{and}$$

$$\mathbf{a} \sum_{i=1}^{n} x_i \;\mathbf{+}\; \mathbf{b} \sum_{i=1}^{n} x_i^2 \;\mathbf{=}\; \sum_{i=1}^{n} x_i y_i$$

These equations are called the normal equations. Solving these two equations simultaneously we can get the values of a and b..

In the case of a quadratic polynomial the normal equations are given by

$$na_0 + a_1 \sum x_i + a_2 \sum x_i^2 = \sum y_i$$

$$a_0 \sum x_i + a_1 \sum x_i^2 + a_2 \sum x_i^3 = \sum x_i y_i$$

$$a_0 \sum x_i^2 + a_1 \sum x_i^3 + a_2 \sum x_i^4 = \sum x_i^2 y_i$$

Solving these three equations simultaneously one get the values of a_0,a_1 and a_2 and the quadratic equation is fitted as $y= a_0+a_1x+a_2x^2$.

Problems:

1. **Using the method of least squares, fit a straight line of the form y=a+bx to the given data**

x	1	2	3	4
y	1.7	1.8	2.3	3.2

Solution: In this case n = 4

$$\sum x_i = 1 + 2 + 3 + 4 = 10$$

$$\sum y_i = 1.7 + 1.8 + 2.3 + 3.2 = 9$$

$$\sum x_i^2 = 1 + 4 + 9 + 16 = 30$$

$\sum x_i y_i =$ (1x1.7) + (2x1.8) + (3x2.3) + (4x32) = 25

The normal equations are given by

$4a + 10b = 9$

$10a + 30b = 25$

Solving for a and b, we get

$a = 1, \quad b = ½$

The linear equation fitted to the given data is

$y = 1 + (½)x$

2**. Fit a quadratic curve from the following data using the principle of least squares.**

x	0	1	2	3	4
y	1	1.8	1.3	2.5	6.3

The normal equations are

$$na_0 + a_1 \sum x_i + a_2 \sum x_i^2 = \sum y_i$$

$$a_0 \sum x_i + a_1 \sum x_i^2 + a_2 \sum x_i^3 = \sum x_i y_i$$

$$a_0 \sum x_i^2 + a_1 \sum x_i^3 + a_2 \sum x_i^4 = \sum x_i^2 yi$$

Here number of data n=5

$\sum x_i =$ 0 + 1 + 2 + 3 + 4 = 10

$\sum x_i y_i =$ (0x1) + (1x1.8) + (2x1.3) + (3x2.5) + (4x6.3) = 37.1

$\sum x_i^2 =$ 0 + 1 + 4 + 9 + 16 = 30

$\sum x_i^3 =$ 0 + 1 + 8 + 27 + 64 = 100

$\sum x_i^4 = \quad 0 + 1 + 16 + 81 + 256 = 354$

$\sum x_i^2 y_i = \quad 0 + (1x1.8) + (4x1.3) + (9x2.5) + (16x6.3) = 103.3$

With these substitutions equation (2) becomes

$5a_0 + 10a_1 + 30a_2 = 12.9$

$10a_0 + 30a_1 + 100a_2 = 37.1$

$30a_0 + 100a_1 + 354a_2 = 103.3$

Solving these we get

$a_0 = 1.42, \quad a_1 = -1.07, \quad a_2 = 0.55$

The fitted quadratic curve becomes

$$Y = 1.42 - 1.07\,x + 0.55\,x^2$$

UNIT IV

Numerical Differentiation

Numerical differentiation is a process by which one can evaluate the approximate numerical value of the derivative of a function at some assigned value of the independent variable, by using the set of given values of that function. To solve the problem of differentiation, we first approximate the function by an interpolation formula and then differentiate it as many times as .....

In case the given argument values are equally spaced, we represent the function by Newton-Gregory formula. If the derivative of the function is to be evaluated at a point near the beginning of the tabular values, we use Newton-Gregory forward formula. If the derivative of the function is to be evaluated at a point near the end of the tabular values, we use Newton- Gregory backward formula.

Newton's forward difference formula to find numerical differentiation

We are given with (n+1) ordered pairs (x_i, $\mathbf{y_i}$) $i = 0, 1, 2, \ldots n$. We want to find the derivative of $y = f(x)$ passing through the (n+1) points, at a point nearer to the starting value $x = x_0$.

Newton's forward difference interpolation formula is

$$y(x_0 + uh) = y_u = y_0 + u\,\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \ldots\ldots \quad (1)$$

Where y (x) is a polynomial of degree n in x and $u = (x - x_0)/h$

Differentiating y (x) w.r.t. x,

$$dy/dx = (dy/du)(du/dx) = 1/h\ (dy/du)$$

$$= 1/h\{ [\,\Delta y_0 + [(2u-1)/2]\,\Delta^2 y_0 + [(3u^2 - 6u + 2)/6]\,\Delta^3 y_0 + [(4u^3 - 18u^2 + 22u - 6)/24]\,\Delta^4 y_0 + \ldots\ldots\} \quad (2)$$

when $x = x_0$, i.e., $u = 0$, equation (2) can be written as

$$(dy/dx)_{x=x_0} =, (dy/dx)_{u=0}$$

$$= 1/h\,[\Delta y_0 - 1/2\,\Delta^2 y_0 + 1/3\,\Delta^3 y_0 - 1/4\,\Delta^4 y_0 + \ldots] \qquad \ldots (3)$$

Differentiating (2) again w.r.t. x

$$d^2y/dx^2 = d/du\,(dy/dx)\,.(du/dx)$$

$$= d/du\,(dy/dx)\,.\,1/h$$

$$= 1/h^2\,[\Delta^2 y_0 + (u-1)\,\Delta^3 y_0 + (6u^2 - 18u + 11)/12\,\Delta^4 y_0 + \ldots \qquad (4)$$

Equation (4) gives the second derivative value of y with respect to x.

Setting $x = x_0$ i.e., $u = 0$ in (4)

$$(d^2y/dx^2)_{x=x0} = 1/h^2\,[\Delta^2 y_0 - \Delta^3 y_0 + (11/12)\,\Delta^4 y_0 + \ldots] \qquad \ldots (5)$$

Therefore, the Newton's forward difference formula to evaluate numerical differentiation is

$$(dy/dx)_{x=x0} = 1/h\,[\Delta y_0 - 1/2\,\Delta^2 y_0 + 1/3\,\Delta^3 y_0 - \ldots]$$

$$(d^2y/dx^2)_{x=x0} = 1/h^2\,[\Delta^2 y_0 - \Delta^3 y_0 + 11/12\,\Delta^4 y_0 + \ldots]$$

Newton's backward difference formula for numerical differentiation

Newton's backward difference interpolation formula is

$$y(x) = y(x_n + vh)$$

$$= y_n + v\,\nabla y_n + \frac{v(v+1)}{2!}\,\nabla^2 y_n + \frac{v(v+1)(v+2)}{3!}\,\nabla^3 y_n + \ldots \qquad (6)$$

Here $v = x - x_n / h$

Differentiate (6) w.r.t. x

$$dy/dx = (dy/dv)\,(dv/dx) = (dy/dv)\,(1/h)$$

$$dy/dx = 1/h\,[\nabla y_n + \frac{2v+1}{2}\,\nabla^2 y_n + \frac{3v^2 + 6v + 2}{6}\,\nabla^3 y_n + \frac{4v^3 + 18v^2 + 22v + 6}{24}\,\nabla^4 y_n + \ldots] \qquad \ldots (7)$$

$$d^2y/dx^2 = 1/h^2\,[\nabla^2 y_n + (v+1)\,\nabla^3 y_n + \frac{6v^2 + 18v + 11}{12}\,\nabla^4 y_n + \ldots] \qquad \ldots (8)$$

Equations (7) and (8) give the first second derivative at general x.

Setting $x = x_n$ or $v = 0$ in (7) and (8) we get

$$dy/dx = 1/h\,[\nabla y_n + 1/2\ \nabla^2 y_n + 1/3\ \nabla^3 y_n + 1/4\ \nabla^4 y_n + \ldots] \quad \ldots (9)$$

$$d^2y/dx^2 = 1/h^2\,[\nabla^2 y_n + \nabla^3 y_n + 11/12\ \nabla^4 y_n + \ldots\ ] \quad \ldots (10)$$

Equations (9) and (10) are the Newton's backward difference formulae to evaluate numerical differentiation.

Problems

1. Find the first two derivatives of $(x)^{1/3}$ at $x = 50$ and $y = 56$ given in the table below

x	50	51	52	53	54	55	56
$y = x^{1/3}$	3.6840	3.7084	3.7325	3.7563	3.7798	3.8030	3.8259

Solution:

Since we require $f^!(x)$ at $x = 50$, we use Newton's forward difference formula and to get $f^!(x)$ at $x = 56$ we have to use Newton's backward difference formula

Difference Table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
50	3.6840			
		0.0244		
51	3.7084		- 0.0003	
		0.0241		0
52	3.7325		- 0.0003	
		0.0238		0
53	3.7563		- 0.0003	
		0.0235		0
54	3.7798		- 0.0003	
		0.0232		0
55	3.8030		- 0.0003	
		0.0229		
56	3.8259			

By Newton's forward difference formula,

$$\left(\frac{dy}{dx}\right)_{x=x_0} = \left(\frac{dy}{dx}\right)_{u=0}$$

$$= 1/h\,[\,\Delta y_0 - 1/2\,\Delta^2 y_0 + 1/3\,\Delta^3 y_0 \;\ldots\,]$$

$$= 1/1\,[\,0.0244 - 1/2\,(-\,0.0003) + 1/3\,(0)\,]$$

$$= \mathbf{0.02455}$$

$$\left(\frac{d^2y}{d^2x}\right)_{x=50} = 1/h^2\,[\,\Delta^2 y_0 - \Delta^3 y_0 + \ldots\,]$$

$$= 1\,[\,-\,0.0003\,]$$

$$= \mathbf{-\,0.0003}$$

By Newton's backward difference formula,

$$\left(\frac{dy}{dx}\right)_{x=x_n} = \left(\frac{dy}{dx}\right)_{v=v_0}$$

$$= 1/h\,[\,\nabla y_n - 1/2\,\nabla^2 y_n + 1/3\,\nabla^3 y_n \;\ldots\,]$$

$$= 1/1\,[\,0.0229 + 1/2\,(-\,0.0003) + 1/3\,(0)\,]$$

$$= \mathbf{0.02275}$$

$$\left(\frac{d^2y}{d^2x}\right)_{x=56} = 1/h^2\,[\,\nabla^2 y_n + \nabla^3 y_n + \ldots\,]$$

$$= 1\,[\,-\,0.0003\,]$$

$$= \mathbf{-\,0.0003}$$

2. The population of a town is given below. Find the rate of growth of the population in the years 1931, 1941, 1961 and1971.

Year x	1931	1941	1951	1961	1971
Population in thousands y	40.62	60.80	79.95	103.56	132.65

Solution

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1931	40.62				
		20.18			
1941	60.80		-1.03		
		19.15		5.49	
1951	79.95		4.46		-4.47
		23.61		1.02	
1961	103.56		5.48		
		29.09			
1971	132.65				

(i) To get f^1 (1931) and f^1(1941) we use forward formula.

x_0 = 1931, x_1 = 1941, ….

u = x – x_0 / h Here x_0 = 1931 Corresponding to u = 0

By Newton's forward difference formula,

$$\left(\frac{dy}{dx}\right)_{x=1931} = \left(\frac{dy}{dx}\right)_{u=0}$$

$$= \frac{1}{h}\left[\Delta y_0 - 1/2\,\Delta^2 y_0 + 1/3\,\Delta^3 y_0 - 1/4\,\Delta^4 y_0 + \ldots \right]$$

$$= 1/10\,[\,(20.18) - 1/2(-1.03) + 1/3(5.49) - 1/4(-4.47)\,]$$

$$= 1/10\,[\,20.18 + 0.515 + 1.83 + 1.1175\,]$$

= **2.36425**

(ii) when x = 1941, we get u = x – x_0 / h = (1941 – 1931)/10 = 1

Putting u = 1 in the formula given below

$$\frac{dy}{dx} = 1/h\,[\,\Delta y_0 + [(2u-1)/2]\,\Delta^2 y_0 + [(3u^2 - 6u + 2)/6]\,\Delta^3 y_0 +$$

$$(4u^3 - 18u^2 + 22u - 6)/24]\,\Delta^4 y_0 + \ldots\ldots$$

We get

$$\left(\frac{dy}{dx}\right)_{u=1} = 1/10\,[\,(20.18) + 1/2(-1.03) - 1/6(5.49) + 1/12(-4.47)]$$

$$= 1/10\,[\,20.18 - 0.515 - 0.915 - 0.3725]$$

$$= \mathbf{1.83775}$$

(iii) To get $f^!$ (1971), we use the formula

$$\left(\frac{dy}{dx}\right)_{x=x0} = 1/h\,[\nabla y_n + 1/2\ \nabla^2 y_n + 1/3\,\nabla^3 y_n + 1/4\ \nabla^4 y_n + \ldots]$$

$$\left(\frac{dy}{dx}\right)_{1971} = 1/10\,[\,29.09 + 1/2(5.48) + 1/3(1.02) + 1/4\,(-4.47)\,]$$

$$= 1/10\,[31.10525]$$

$$= \mathbf{3.10525}$$

(iv) To get $f^!$ (1961), we use

$$v = x - x_n /h = (1961 - 1971)/10 = -1$$

$$\left(\frac{dy}{dx}\right)_{x=1961} = 1/h\,[\nabla y_n + \frac{2v+1}{2}\nabla^2 y_n + \frac{3v^2 + 6v + 2}{6}\nabla^3 y_n + \ldots]$$

$$= 1/10\,[\,29.09 - 1/2(5.48) - 1/6(1.02) - 1/4\,(-4.47)\,]$$

$$= 1/10\,[29.09 - 2.74 - 0.17 + 0.3725\,]$$

$$= \mathbf{2.65525}$$

3. Find the first and second derivatives of the function tabulated below at the point $x = 1.5$

x	1.5	2.0	2.5	3.0	3.5	4.0
f (x)	3.375	7.0	13.625	24.0	38.875	59.0

Solution

The difference table is as follows:

x	y = f(x)	Δ y	Δ^2 y	Δ^3 y	Δ^4 y
1.5 (x_0)	3.375 (y_0)				
		3.625 $=\Delta y_0$			
2.0	7.0		3.0 $=\Delta^2 y_0$		
		6.625		0.75 $\Delta^3 y_0$	
2.5	13.625		3.75		0 $=\Delta^4 y_0$
		10.375		0.75	
3.0	24.0		4.5		0
		14.875		0.75	
3.5	38.875		5.25		
		20.125			
4.0	59.0				

Here we have to find the derivative at the point $x = 1.5$ which is the initial value of the table. Therefore by Newton's forward difference formula for derivatives at

$x = x_0$, we have

$$f'(x_0) = 1/h\,[\Delta y_0 - 1/2\,\Delta^2 y_0 + 1/3\,\Delta^3 y_0 - \ldots]$$

Here $x_0 = 1.5$, $h = 0.5$

$$f'(1.5) = (1/0.5)\,[\,(3.625) - 1/2\,(3.0) + 1/3\,(0.75)\,]$$

$$\mathbf{f'(1.5) = 4.75}$$

At the point $x = x_0$,

$$f''(x_0) = 1/h^2\,[\,\Delta^2 y_0 - \Delta^3 y_0 + 11/12\,\Delta^4 y_0 + \ldots\,]$$

Here $x_0 = 1.5$, $h = 0.5$

$$f''(1.5) = [1/(0.5)^2]\,[\,(3.0) - (0.75)\,]$$

$$\mathbf{f'(1.5) = 9.0}$$

Numerical Integration

Introduction:

We know that $\int_a^b f(x)dx$ represents the area between y = f (x), x-axis and the ordinates x = a and x = b. This integration is possible only if the function is explicitly given and if it is integrable. We can replace f (x) by an interpolating polynomial P_n (x) and obtain $\int Pn(x)$ dx which is approximately taken as the value for $\int_{x0}^{xn} f(x)dx$.

Newton-cote's formula for Numerical Integration

For equally spaced intervals, we have Newton's forward difference formula as

$$f(x) = y(x_0 + uh) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \ldots\ldots \quad (1)$$

f (x) can be replaced by Newton's interpolating formula in integration.

Here, $u = (x - x_0)/h$ where h is interval of differencing

Since $x_n = x_0 + nh$ and $u = (x - x_0)/h$, we have $(x - x_0)/h = n = u$

$$\int_{x0}^{xn} f(x)dx = \int_{x0}^{x0+nh} f(x)dx$$

$$= \int_{x0}^{x0+nh} P_n(x)\,dx,$$ where P_n (x) is the interpolating polynomial of degree n

$$= \int_0^n \left[y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \ldots \right] hdu$$

Since dx = hdu, and when x = x_0, the lower limit u = 0 and when x = x_0 + nh, the upper limit u = n

$$= h\int_0^n \left[y_0 + u\Delta y_0 + \frac{u^2 - u}{2!}\Delta^2 y_0 + \frac{u^3 - 3u^2 + 2u}{3!}\Delta^3 y_0 + \ldots \right] du$$

$$= h\left\{ y_0(u) + (u^2/2)\Delta y_0 + 1/2\left[(u^3/3) - (u^2/2)\right]\Delta^2 y_0 + 1/6\left[(u^4/4) - u^3 + u^2\right]\Delta^3 y_0 + \ldots \right\}$$

$$\int_{x0}^{xn} f(x)dx = h\left\{ [\, n y_0 + (n^2/2)\Delta y_0 + 1/2\left[(n^3/3) - (n^2/2)\right]\Delta^2 y_0 + 1/6\left[(n^4/4) - n^3 + n^2\right]\Delta^3 y_0 + \ldots \right\} \quad \ldots (2)$$

Equation (2) is called as Newton-Cote's quadrature formula or in general quadrature formula. Various values for n yield number of special formulae.

Trapezoidal rule

When n = 1, in the quadrature formula (i.e. there are only two paired values and the interpolating polynomial is linear) we get

$\int_{x0}^{x0+h} f\ (x)dx$ = h [y_0 + 1/2 Δ y_0] since other differences do not exist when n=1

= h [y_0 + 1/2 ($y_1 - y_0$)]

= h/2 ($y_0 + y_1$)

$\int_{x0}^{xn} f\ (x)dx$ = $\int_{x0}^{x0+nh} f\ (x)dx$

= $\int_{x0}^{x0+h} f\ (x)dx + \int_{x0}^{x0+2h} f\ (x)dx + ... + \int_{x0}^{x0+nh} f\ (x)dx$

= h/2 ($y_0 + y_1$) + h/2 ($y_1 + y_2$) + … + h/2 ($y_{n-1} + y_n$)

= h/2 [($y_0 + y_n$) + 2 ($y_1 + y_2 + y_3 + ... + y_{n-1}$)]

= h/2 [(sum of the first and last ordinates) +

2(sum of the remaining ordinates)]

This is known as Trapezoidal Rule.

This method is very simple for calculation purposes of numerical integration. The error in this case is significant. The accuracy of the result can be improved by increasing the number of intervals and decreasing the value of h.

Truncation error in Trapezoidal rule

In the neighbourhood of x = x_0, we can expand y = f (x) by Taylor series in powers of x – x_0. That is,

$$y\ (x) = y_0 + \frac{(x - x_0)}{1!} y_0' + \frac{(x - x_0)^2}{2!} y_0'' + ... \qquad ... \qquad (1)$$

Where $y_0' = [y'(x)]_{x=x0}$

$$\int_{x0}^{x1} y\,dx = \int_{x0}^{x1} \left[y_0 + \frac{(x - x_0)}{1!} y_0' + \frac{(x - x_0)^2}{2!} y_0'' + \ldots.. \right] dx$$

$$= \left[y_0 x + \overline{(x - x_0)}\, y_0' + (x - x_0)^2 y_0'' + \ldots. \right]$$ with upper limit x_1 and lower limit x_0

$$= y_0 (x_1 - x_0) + \frac{(x_1 - x_0)^2}{2!} y_0' + \frac{(x_1 - x_0)^3}{3!} y_0'' + \ldots.$$

$$= h\, y_0 + (h^2/2!)\, y_0' + (h^3/3!)\, y_0'' + \ldots \qquad \ldots \quad (2)$$

where h is the equal interval length.

Also $\int_{x0}^{x1} y\,dx = h/2\,(y_0 + y_1)$ = area of the first trapezium = A_0 … (3)

Putting $x = x_1$ in (1)

$$y(x_1) = y_1 = y_0 + \frac{(x_1 - x_0)}{1!} y_0' + \frac{(x_1 - x_0)^2}{2!} y_0'' + \ldots..$$

i.e. $$y_1 = y_0 + (h/1!)\, y_0' + (h^2/2!)\, y_0'' + \ldots \qquad \ldots \quad (4)$$

$$A_0 = h/2\,[\,y_0 + y_0 + (h/1!)\, y_0' + (h^2/2!)\, y_0'' + \ldots\,]$$ using (4) in (3)

$$= h\, y_0 + (h^2/2)\, y_0' + (h^3/2\text{x}2!)\, y_0'' + \ldots$$

Subtracting A_0 value from (2)

$$\int_{x0}^{x1} y\,dx - A_0 = h^3 y_0''\,[\,(1/3!) - (1/2\text{x}2!)\,] + \ldots.$$

$$= -(1/12)\, h^3 y_0'' + \ldots.$$

Therefore the error in the first interval (x_0, x_1) is $-(1/12)\, h^3 y_0''$ (neglecting other terms)

Similarly the error in the i^{th} interval $= -(1/12)\, h^3 y_{i-1}''$

Therefore, the total error is $E = -1/12\, h^3\, (y_0'' + y_1'' + y_2'' + \ldots. + y_{n-1}'')$

$|E| < nh^3/12\,.\,M$, where M is the maximum value of

$|y_0''|\ \ |y_1''|\ \ |y_2''|\ \ \ldots..$

$|E| < (b - a)\, h^2/12\,.\,M$

If the interval is (a, b) and $h = (b - a)/n$

Hence, the error in the trapezoidal rule is of the order h^2

Simpson's one-third rule

Setting $n = 2$ in Newton-cote's quadrature formula, we have

$$\int_{x0}^{x0+h} f(x)dx = h\,[\,2\,y_0 + 4/2\,\Delta\,y_0 + 1/2(\,8/3 - 4/2)\,\Delta^2\,y_0\,]$$

(since other terms vanish)

$$= h\,[\,2y_0 + 2\,(\,y_1 - y_0\,) + 1/3\,(\,E - 1\,)^2\,y_0\,]$$

$$= h\,[\,2y_0 + 2y_1 - 2y_0 + 1/3\,(y_2 - 2y_1 + y_0\,)\,]$$

$$= h\,[\,1/3\,y_2 + 4/3\,y_1 + 1/3\,y_0\,]$$

$$= h/3\,[\,y_2 + 4y_1 + y_0]$$

Similarly, $\int_{x2}^{x4} f(x)dx = h/3\,[\,y_2 + 4y_3 + y_4]$

$$\int_{xi}^{xi+2} f(x)dx = h/3\,[\,y_i + 4y_{i+1} + y_{i+2}]$$

If n is an even integer, last integral will be

$$\int_{xn}^{xn+2} f(x)dx = h/3\,[\,y_{n-2} + 4y_{n-1} + y_n]$$

Adding all these integrals, if n is an even positive integer, that is, the number of ordinates $y_0, y_1, \ldots. Y_n$ is odd, we hae

$$\int_{x0}^{xn} f(x)dx = \int_{x0}^{x2} f(x)dx + \int_{x2}^{x4} f(x)dx + \ldots. + \int_{xn-2}^{xn} f(x)dx$$

$$= h/3\,[\,(y_0 + 4y_1 + y_2) + (y_2 + 4y_3 + y_4) + \ldots + (y_{n-2} + 4y_{n-1} + y_n)\,]$$

$$= h/3\,[\,(\,y_0 + y_n\,) + 2\,(\,y_2 + y_4 + \ldots\,) + 4\,(\,y_1 + y_3 + \ldots\,)]$$

= h/3 [sum of the first and last ordinates

+ 2 (sum of remaining odd ordinates)

+ 4 (sum of the even ordinates)]

Note: Though y_2 has suffix even, it is the third ordinate (odd)

Simpson's three-eighths rule

Putting n = 3 in Newton-cote's formula we get

$$\int_{x0}^{x3} f\,(x)dx = h\,[\,3\,y_0 + 9/2\,\Delta\,y_0 + 1/2(\,9/2)\,\Delta^2\,y_0 + 1/6(\,81/4 - 27 + 9)\,\Delta^3 y_0\,]$$

$$= h\,[\,3\,y_0 + 9/2\,(\,y_1 - y_0\,) + 9/4(\,E - 1\,)^2\,y_0 + 3/8(\,E - 1)^3\,y_0\,]$$

$$= h\,[\,3\,y_0 + (9/2)\,y_1 - (9/2)\,y_0 + 9/4\,(\,y_2 - 2y_1 + y_0) +$$

$$3/8\,(\,y_3 - 3y_2 + 3y_1 - y_0\,)]$$

$$= 3h/8\,[\,y_3 + 3y_2 + 3y_1 + y_0\,]$$

If n is a multiple of 3,

$$\int_{x0}^{x0+nh} f\,(x)dx = \int_{x0}^{x0+3h} f\,(x)dx + \int_{x0+3h}^{x0+6h} f\,(x)dx + \ldots + \int_{x0+(n-3)h}^{x0+nh} f\,(x)dx$$

$$= 3h/8\,[\,(y_0 + 3y_1 + 3y_2 + y_3) + (y_3 + 3y_4 + 3y_5 + y_6) + \ldots..$$

$$+ (y_{n-3} + 3y_{n-2} + 3y_{n-1} + y_n)\,]$$

$$= 3h/8\,[\,(\,y_0 + y_n\,) + 3\,(\,y_1 + y_2 + y_4 + y_5 + \ldots. + y_{n-1}\,)$$

$$+ 2\,(\,y_3 + y_6 + y_9 + \ldots + y_n\,)\,]$$

The above equation is called Simpson's three-eighths rule which is applicable only when n is a multiple of 3.

Problems

1. Evaluate $\int_{-3}^{+3} x^4\,dx$ by using (1) Trapezoidal rule (2) Simpson's rule. Verify your results by actual integration.

Solution

Here $y\,(x) = x^4$. Interval length (b – a) = 6, so, we divide equal intervals with h = 6/6 = 1, we form below the table

x	-3	-2	-1	0	1	2	3
y	81	16	1	0	1	16	81

By Trapezoidal rule,

$\int_{-3}^{3} y\,dx$ = h/2 [(sum of the first and last ordinates) + 2(sum of the remaining ordinates)]

= 1/2 [(81 + 81) + 2 (16 + 1 + 0 + 1 + 16)]

= 115

By Simpson's one – third rule (since number of ordinate is odd)

$\int_{-3}^{3} y\,dx$ = 1/3 [(81 + 81) + 2 (1 + 1) + 4 (16 + 0 + 16)]

= 98

Since n = 6, (multiple of three), we can use Simpson's three-eighths rule. By this rule

$\int_{-3}^{3} y\,dx$ = 3/8 [(81 + 81) + 3 (16 + 1 + 1 + 16) + 2 (0)]

= 99

By actual integration

$\int_{-3}^{3}$ x^4 dx = 2 x $\left[x^5/5\right]$ with upper limit 3 and lower limit 0.

= 2 x 243 / 5

= 97.2

From the results obtained by various methods, we see that Simpson's rule gives better result than Trapezoidal rule (It is true in general)

2. Evaluate $\int_{0}^{1} dx$ / $1+x^2$, using Trapezoidal rule and h = 0.2. Hence obtain an approximate value of π.

Solution

Let y (x) = $1/(1+x^2)$

Interval is (1 – 0) = 1 Since the value of y are calculated as points using h = 0.2

x	0	0.2	0.4	0.6	0.8	1.0
$y = 1/(1+x^2)$	1	0.96154	0.86207	0.73529	0.60976	0.50000

(1) By Trapezoidal rule

$\int_0^1 dx \,/\, 1+x^2$ = h/2 [(y_0 + y_n) + 2 (y_1 + y_2 + ….+ y_{n-1})]

= 0.2/2 [(1 + 0.5) + 2 (0.96154 + 0.086207 + 0.73529 + 0.60976)

= (0.1) [1.5 + 6.33732]

= 0.783732

By actual integration,

$\int_0^1 dx \,/\, 1+x^2$ = $(\tan^{-1} x)_0^1$ = $\pi/4$

$\pi/4$ = 0.783732.

Hence π = (4X0.783732) = 3.13493

3. From the following table, find the area bounded by the f (x) and the x-axis from x = 7.47 to x = 7.52

x	7.47	7.48	7.49	7.50	7.51	7.52
Y= f (x)	1.93	1.95	1.98	2.01	2.03	2.06

Solution

Using Trapezoidal rule we can write the area as the integral of f(x).

$\int_{7.47}^{7.52} f\,(x)dx$ = 0.01/2 [(1.93 + 2.06)+2(1.95 + 1.98 + 2.01 + 2.03)]

= 0.09965

4. Evaluate I = $\int_0^6$ dx / (1 + x) using (1) Trapezoidal rule (2) Simpson's rule. Verify your results by actual integration.

Solution

Take the number of intervals as 6. Therefore h = 6 – 0 / 6 = 1

x	0	1	2	3	4	5	6
y=1/(1+x)	1	1/2	1/3	1/4	1/5	1/6	1/7

(1) By Trapezoidal rule,

$\int_0^6 dx/1+x$ = 1/2 [(1 + 1/7) + 2 (1/2 + 1/3 + 1/4 + 1/5 + 1/6)]

= **2.02142857**

(2) By Simpson's one – third rule

I = 1/3 [(1 + 1/7) + 2 (1/3 + 1/5) + 4 (1/2 + 1/4 + 1/6)]

= 1/3 [1 + 1/7 + 16/15 + 22/6

= **1.95873016**

(3) By Simpson's three-eighths rule.

I = 3/8 [(1 + 1/7) + 3 (1/2 + 1/3 + 1/5 + 1/6) + 2 (1/4)]

= **1.96607143**

(4) By actual integration

$\int_0^6 dx/1+x$ = $[\log (1 + x)]_0^6$

= $\log_e 7$

= **1.94591015**

5. Evaluate $\int_0^1 e^{-x2}$ dx by dividing the range of integration into equal parts using Simpson's one-third rule.

Solution

Here the length of the interval is h = 1- 0/4 = 0.25. The values of the function y = e^{-x2} for each point of subdivision are given below

x	0	0.25	0.5	0.75	1
e^{-x2}	1 y_0	0.9394 y_1	0.7788 y_2	0.5698 y_3	0.3678 y_4

By Simpson's rule we have

$$\int_0^1 e^{-x^2}\,dx = h/3\,[\,(y_0 + y_4) + 2\,y_2 + 4\,(y_1 + y_3)\,]$$

$$= 0.25/3\,[\,1.3678 + 1.5576 + 6.0368\,]$$

$$\int_0^1 e^{-x^2}\,dx = \mathbf{0.7468}$$

Gauss Quadrature formula

Carl Frederich Gauss approached the problem of numerical integration in a different way. Instead of finding the area under the given curve, he tried to evaluate the function at some points along with the abscissa. Here the values of abscissa are not equal. Then apply certain weight to the evaluated function.

Thus for Gauss two point formula

$$\int_a^b f(x)dx = \int_{-1}^1 f(t)dt$$

$$= \omega_1 f(t_1) + \omega_2 f(t_2) \qquad \ldots \qquad (1)$$

The function f (t) is evaluated at t_1 and t_2. ω_1 and ω_2 are the weights given to the two functions.

The basic methodology is explained as given below for **Gauss Two Point Formula.**

Gauss Two Point Formula

First one has to change the interval (a,b) to (-1, 1) by using the following transformation equation

$X = [(a + b)/2] + [(b – a)/2]\,t$

Thus the independent variable 'x' is changed to 't'.

Then we use an interpolation formula which will give the true value of the integral at certain points. Here the interpolation points are t_1 and t_2.

In equation (1), we want to find the four unknown quantities ω_1, ω_2 and t_1 t_2. So we need four algebraic equations to solve it. Let the equation (1) be exact for

$f(t) = 1$

$f(t) = t$

$f(t) = t^2$ and

$f(t) = t^3$

when $f(1) = 1$ we get

$$\int_{-1}^{1} 1\,dt = 2 = \omega_1 + \omega_2 \qquad \ldots \quad (2)$$

[since $f(t_1) = f(t_2) = 1$]

When $f(t) = t$

$$\int_{-1}^{1} t\,dt = \left[t^2/2 \right]_{-1}^{1} = 0 = \omega_1 t_1 + \omega_2 t_2 \qquad \ldots \quad (3)$$

When $f(t) = t^2$ we get

$$\int_{-1}^{1} t^2\,dt = \left[t^3/3 \right]_{-1}^{1} = 2/3 = \omega_1 t_1^2 + \omega_2 t_2^2 \qquad \ldots \quad (4)$$

when $f(t) = t^3$

$$\int_{-1}^{1} t^3\,dt = \left[t^4/4 \right]_{-1}^{1} = 0 = \omega_1 t_1^3 + \omega_2 t_2^3 \qquad \ldots \quad (5)$$

This set of equations (2), (3), (4) and (5) can be solved as follows

From (3) we get

$$\omega_1 t_1 = -\omega_2 t_2 \qquad \ldots \quad (6)$$

From (5) we get

$$\omega_1 t_1^3 = -\omega_2 t_2^3 \qquad \ldots \quad (7)$$

From (6) and (7) we get

$t_1 = -t_2$

$\omega_1 = \omega_2 = 1$

From (4) we get

$t_1^2 + t_2^2 = 2/3$

$t_1 = 1/\sqrt{3}$

t_2 = - $1/\sqrt{3}$

From equation (1) we get

$$I = \int_{-1}^{1} f(t)dt = \omega_1 f(t_1) + \omega_2 f(t_2)$$

$$I = f(1/\sqrt{3}) + f(-1/\sqrt{3}) \quad \ldots \quad (A)$$

[since $\omega_1 = \omega_2 = 1$]

Problems

1. Evaluate $\int_1^2 dx/x$ using Gauss two point formula.

Solution

Transform the variable x to t by the transformation

$X = [(a + b)/2] + [(b – a)/2] t$

$= [(1 + 2)/2] + [(2 - 1)/2)] t$

$X = 3/2 + t/2 = (3 + t)/2$

i.e. $dx = dt/2$

Therefore $I = \int_1^2 dx/x = \int_{-1}^{1} \frac{2}{3+t}\frac{dt}{2}$

$$= \int_{-1}^{1} \frac{dt}{3+t}$$

Here $f(t) = \frac{1}{3+t}$

$f(1/\sqrt{3}) = 1/(3 + \sqrt{3}) = 0.2795$

$f(-1/\sqrt{3}) = 1/(3 - \sqrt{3}) = 0.41288$

$I = f(1/\sqrt{3}) + f(-1/\sqrt{3})$

I = 0.6923

2. Evaluate $\int_1^2 dx/$ (1+x^3) using Gauss 2 point formula.

Solution

Transform the variable x to t by the transformation equation

X = [(a + b)/2] + [(b – a)/2] t

X = 3/2 + t/2 = (3 + t) / 2

and dx = dt/2

Therefore I = $\int_1^2 dx/$ (1+x^3)

$$= \int_{-1}^{1} \frac{1}{1+((3+t)/2)3} \frac{dt}{2}$$

= 4 $\int_{-1}^{1}$ dt/ 8 + (3+t)^3

= 4 [f (1/√3) + f (- 1/√3)]

Here f (t) = 1 / [8 + (3 +t)^3]

f (1/√3) = 1/[8 + (3 + 1/√3)^3 = 0.0185

f (-1/√3) = 1/[8 + (3 - 1/√3)^3 = 0.045

I = 4 [f (1/√3) + f (- 1/√3)]

= 4 [0.0185 + 0.045]

I = 0.254

Gauss three point formula

$\int_a^b f(x)dx$ = $\int_{-1}^1 f(t)dt$

Where the interval (a, b) is changed into (-1, 1) by the transformation

X = [(b + a)/2] + [(b – a)/2] t

Then

$\int_{-1}^1 f(t)dt$ = A_1 f (t_1) + A_2 f (t_2) + A_3 f (t_3)

Where

A_1 = A_3 = 0.5555

A_2 = 0.8888

t_1 = - 0.7745

t_2 = 0

t_3 = 0.7745

Problem

1. Evaluate $\int_1^2 dx/x$ using Gauss three point formula.

Solution

Transform the variable x to t by the transformation

X = [(b + a)/2] + [(b – a)/2] t

= [(1 + 2)/2] + [(2 -1)/2] t

x = 3/2 + t/2 = (3 + t)/ 2 and dx = dt/2

Therefore I = $\int_1^2 dx/x$ = $\int_{-1}^{1}$ f (t) dt

= A_1 f (t_1) + A_2 f (t_2) + A_3 f (t_3)

A_1 = A_3 = 0.5555

A_2 = 0.8888

In this problem f(x) = $\frac{1}{x}$ and f(t)= 1/(3+t)

f (t_1) = f (-0.7745)

= 1/ (3 – 0.7745) = 0.4493

f (t_2) = f (0)

= 1/ 3 = 0.3333

f (t_3) = f (0.7745)

= 1/ (3 + 0.7745) = 0.2649

Substituting the values of A_1 , A_2, A_3 and f (t_1), f (t_2), f (t_3) in the formula, we get

$$I = 0.5555\ (0.4493) + 0.8888\ (0.3333) + 0.5555\ (0.2649)$$

$$\mathbf{I = 0.6929}$$

2. Evaluate $\int_{0.2}^{1.5} e^{-x^2}\,dx$ by using the three point Gaussian Quadrature formula.

Solution

Transform the variable from x to t by the transformation

$$X = [(b + a)/2] + [(b - a)/2]\ t$$

where $a = 0.2 \qquad b = 1.5$

$$= (1.7/2) + (1.3/2)\ t$$

i.e. $x = (1.7 + 1.3t)/2$ and $dx = 1.3\ t/2 = 0.65\ t$

$$I = \int_{0.2}^{1.5} e^{-x^2}\,dx$$

$$= \int_{-1}^{1} e^{-[(1.7 + 1.3t)/2]^2}\,(0.65)dt$$

$$= 0.65 \int_{-1}^{1} e^{-[(1.7 + 1.3t)/2]^2}\,dt$$

Using Gauss three point formula we can write

$$I = 0.65\ [A_1 f(t_1) + A_2 f(t_2) + A_3 f(t_3)]$$

Where $f(t) = e^{-[(1.7 + 1.3t)/2]^2}$

$$A_1 = A_3 = 0.5555$$

$$A_2 = 0.8888$$

$$f(t_1) = f(-0.7745) = e^{-[(1.7 + 1.3(-0.7745))/2]^2}$$

$$= 0.8868$$

$$f(t_2) = f(0) = e^{-[(1.7 + 1.3(0))/2]^2}$$

$$= 0.48555$$

$$f(t_3) = f(0.7745) = e^{-[(1.7 + 1.3(0.7745))/2]^2}$$

$$= 0.16013$$

Substituting the values of A_1 , A_2, A_3 and f (t_1), f (t_2), f (t_3) in the formula, we get

I = 0.5555 (0.8868) + 0.8888 (0.4855) + 0.5555 (0.16013)

= 0.4926 + 0.4315 + 0.08895

I = 1.01307

3. Evaluate $\int_0^1$ dx/ 1+x^2, using Gauss 3 point formula

Solution

Transform the variable from x to t by the transformation

X = [(b + a)/2] + [(b – a)/2] t

where a = 0 b = 1

x = (1/2) + (t/2)

i.e. x = (t + 1)/2 when x = 0, t = -1

x = 1, t = 1

dx = dt/2 = 0. 5t

$$I = \int_0^1 \frac{dx}{(1+x2)}$$

$$= \int_{-1}^1 [\,1/\ (1 + (\ (\ t + 1)/2)^{2)}\ ]*\ dt/2$$

= 2 $\int_{-1}^1$ dt/ 4 + (t + 1)^2 using Gauss three point formula we get

I = 2 [A_1 f (t_1) + A_2 f (t_2) + A_3 f (t_3)]

Where f (t) = 1/ 4 + $(t+1)^2$

A_1 = A_3 = 0.5555

A_2 = 0.8888

f (t_1) = f (- 0.7745) = 1/ 4 + (- 0.7745 + 1)^2

= 0.2468

f (t_2) = f (0) = 1/ 4 + 1 = 0.2

$f(t_3) = f(0.7745) = 1/4 + (0.7745 + 1)^2$

$= 0.13988$

Substituting the values of A_1 , A_2, A_3 and $f(t_1)$, $f(t_2)$, $f(t_3)$ in the formula, we get

$I = 2 [0.5555 (0.2468) + 0.8888 (0.2) + 0.5555 (0.13988)]$

$= 2 [0.39256]$

$I = 0.78512$

Unit V Initial value problems

Solution of first order differential equations – Taylor series method

Initial value problem is an ordinary differential equation given along with a specified initial value of the unknown function at a given point in the domain of the solution. In other words, initial value problem is defined as the problem of finding a function y of x when we know its derivative and its value y_0 at a particular point x_0.

A first order differential equation is given by $\frac{dy}{dx} = y' = f(x,y)$ with the condition that $y(x_0)=y_0$. The approximate solution of a first order differential equation is given by

$(y_m - y_{m-1}) = f(x_m,y_m)$. (or) $y_m = y_{m-1} + f(x_m,y_m)$ and this method is called as single step method. Taylor series method is one such single step method.

Let us consider a first order differential equation $\frac{dy}{dx} = f(x,y)$ with the initial condition $y(x_0)=y_0$. We can expand $y(x)$ about a point x_0 in Taylor series as

$y(x) = y(x_0) + (x-x_0)/1! \; [y'(x)]_{x0} + (x-x_0)^2/2! \; [y''(x)]_{x0} + (x-x_0)^3/3! \; [y'''(x)]_{x0} + \ldots\ldots$

$= y_0 + (x-x_0)/1! \; y'_0 + (x-x_0)^2/2! \; y''_0 + (x-x_0)^3/3! \; y'''x_0 + \ldots\ldots$

When $x=x_0+h =x_1$ we can write,

$y(x_1) = y_0 + h/1! \; y'_0 + h^2/2! \; y''_0 + h^3/3! \; y'''_0 + \ldots..$

In general we can write $\mathbf{y_{m+1} = y_m + h/1! \; y'_m + h^2/2! \; y''_m + h^3/3! \; y'''_m + \ldots..}$

Problems

1. Solve the initial value problem $\frac{dy}{dx} = x^2 - y$ with the initial condition y(0)=1 by Taylor series method and find y(0.1) and y(0.2).

The formula we have to use is $y_1 = y_0 + h/1!\ y'_0 + h^2/2!\ y''_0 + h^3/3!\ y'''_0 + \ldots$

Here $\frac{dy}{dx} = y' = x^2 - y$ and it is given that y=1 when x=0.

Differentiating $y' = x^2 - y$ wrt x we get $y'' = -y'$

Ie. $y'' = -(x^2 - y) = y - x^2$

Here $x_0=0$, $y_0=1$ and $h=(x-x_0) = 0.1$

$y'_0 = x_0^2 - y_0 = 0-1 = -1$

$y''_0 = y_0 - x_0^2 = 1-0 = 1$

$y_1 = y_0 + h/1!\ y'_0 + h^2/2!\ y''_0$

$y(0.1) = y_1 = 1 + (0.1)\,[(-1) + (0.1)^2/2]\,(1) = 1 - 0.1 + \frac{0.01}{2} = 0.905$

Similarly $y_2 = y_1 + h/1!\ y'_1 + h^2/2!\ y''_1$

Here $y_1 = 0.905$, $y'_1 = x_1^2 - y_1 = (0.1)^2 - 0.905 = -0.895$

$y''_1 = y_1 - x_1^2 = 0.905 - (0.1)^2 = 0.895$

Therefore $y_2 = 0.905 + (0.1)(-0.895) + \frac{0.001}{2} 0.895 = 0.82$

Answer : $y_1 = 0.905$ and $y_2 = 0.82$

2. Using Taylor Series method solve the initial value problem $y' = 1 + y^2$ with the initial condition y(0) =0 and find the value y(0.2) and y(0.4).

The formula we have to use is $y_1 = y_0 + h/1!\ y'_0 + h^2/2!\ y''_0$

Here $x_0=0$, $y_0=0$, h=0.2 and $x_1=.2$

$y'_0 = 1 + y_0^2 = 1+0 = 1$

$y''_0 = 2\ y_0\ y'_0 = 0$

$y_1 = y(0.2) = 0 + [\ 0.2\ /1!]\ 1 + [\ (0.2)^2/2!\ ]\ 0 = 0.2$

$y_1 = 0.2$

To find y(0.4) let us use the formula $y_2 = y_1 + h\ /1!\ \ y'_1 + h^2/2!\ \ y''_1$

Here $y'_1 = 1 + y_1^2 = 1+(0.2)^2 = 1.04$ and $y''_1 = 2\ y_1\ y'_1 = 2\ (0.2)\ (1.04) = 0.416$

$y_2 = 0.2+0.2\ (1.04)+[\ (0.2)^2/2]0.416 = 0.4163$

Therefore y(0.4) = 0.4163

Euler's Method

Consider the first order differential equation $\frac{dy}{dx} = y' = f(x,y)$ with the initial condition $y(x_0)=y_0$. Let us assume that the intervals x_0, x_1, x_2, x ,.. are equi distant and at each interval the given function is nearly linear. The problem is to find y_1 at the point $x=x_1$ having known the value of y_0 at $x=x_0$. The equation of tangent for the given function

$\frac{dy}{dx} = y' = f(x,y)$ is written as

$$\frac{y-y0}{x-x0} = (dy/dx)_{x=x0} = f(x_0,y_0)$$

Ie. $y-y_0 = (x-x_0)\ f(x_0,y_0)$

$y = y_0 + (x-x_0)\ f(x_0,y_0)$

When $x_1 = x_0+h$ and $y=y(x_1)=y_1$ we can write

$y_1 = y_0 + h\ \ f(x_0,y_0)$ In a similar manner

$y_2 = y_1 + h\ \ f(x_1,y_1)$ OR in general

$y_{n+1} = y_n + h \ f(x_n,y_n)$ This is the Euler's formula to solve an initial value problem.

Improved Euler's formula

$$y_{n+1} = y_n + \frac{h}{2}\{ f(x_n,y_n) + f(x_{n+h} , y_n+h f(x_n,y_n)) \}$$

Modified Euler's formula

$$y_{n+1} = y_n + h \{ f(x_n+ h/2 , y_n+h/2 f(x_n,y_n)) \}$$

Problems

1.Solve the initial value problem y' = - y with the initial condition y(0)=1 by Euler's method , improved Euler's method and modified Euler's method. And find the value of y(0.01)

Solution

Given $\frac{dy}{dx} = y' = -y$, $x_0=0$, $h=0.01$, $y_0=1$ and $f(x_0,y_0) = -y_0 = -1$

Euler's formula is $y_1 = y_0 + h \ f(x_0,y_0)$

y(0.01) =1 +0.01 (-1) =0.99

Euler's Improved formula is $\mathbf{y_1 = y_0 + \frac{h}{2}\{ f(x_0,y_0) + f(x_{0+h} , y_0+h f(x_0,y_0)) \}}$

$y(0.1)=y_1 = 1+\frac{0.01}{2}[(-1)+(-0.99)] =1-0.00995 = 0.9901$

Euler's modified formula is $y_1 = y_0 + h \{ f(x_0+ h/2 , y_0+h/2 f(x_0,y_0)) \}$

$y(0.1) = y_1 = 1+0.01[-1+\frac{0.01}{2}(-1)] = 0.9901$

2 . Solve the differential equation $\frac{dy}{dx}$ = y' = y-x with the initial condition y(0)=2 evaluate y (0.1) by Euler's methods.

Given: $\frac{dy}{dx} = y' = f(x,y) = y - x$

$x_0 = 0$, $y_0 = 2$, $h = 0.1$, $f(x_0,y_0) = y_0 - x_0 = 2 - 0 = 2$

Euler's formula is $y_1 = y_0 + h\ f(x_0,y_0)$

$y(0.1) = 2 + 0.1\,(2) = 2.2$

Euler's Improved formula is $y_1 = y_0 + \frac{h}{2}\{ f(x_0,y_0) + f(x_{0+h}, y_0 + h\, f(x_0,y_0)) \}$

$y(0.1) = y_1 = 2 + \frac{0.1}{2}[(2-0)+(2.2 - 0.1)] = 2.21$

Euler's modified formula is $y_1 = y_0 + h\,\{ f(x_0 + h/2, y_0 + h/2\ f(x_0,y_0)) \}$

$y(0.1) = y_1 = 2 + 0.1[(2 + \frac{0.1}{2}(2-0) - (0 + \frac{0.1}{2})] = 2.205$

3. Using Euler's method evaluate y(0.2), y(0.4), y(0.6) by solving the equation y' = x + y with the initial condition y(0)=1.

Solution:

Given: $\frac{dy}{dx} = y' = f(x,y) = x + y$

$x_0 = 0$, $y_0 = 1$ $h = 0.2$, $f(x_0,y_0) = y_0 + x_0 = 1 + 0 = 1$

Euler's formula is $y_1 = y_0 + h\ f(x_0,y_0)$

$Y(0.2) = 1 + (0.2)(0+1) = 1.2$ Now

$Y_2 = y_1 + h\ f(x_1,y_1)$ ie. $Y(0.4) = 1.2 + (0.2)(0.2+1.2) = 1.48$

$Y_3 = y_2 + h\ f(x_2,y_2) = 1.48 + 0.2(0.4+1.48) = 1.856$

FOURTH ORDER RUNGE-KUTTA METHOD

This method is most commonly used in practice. Let $dy/dx = f(x,y)$ be a given differential equation to be solved under the condition $y(x_0) = y_0$. Let h be the length of the interval between equidistant values. The first increment in y is computed using the formulae given below.

$k_1 = h f(x_0, y_0)$
$k_2 = h f(x_0 + h/2, y_0 + k_1/2)$
$k_3 = h f(x_0 + h/2, y_0 + k_2/2)$
$k_4 = h f(x_0 + h, y_0 + k_3)$
$\Delta y = 1/6(k_1 + 2k_2 + 2k_3 + k_4)$

Now $x_1 = x_0 + h$, $y_1 = y_0 + \Delta y$

The increment in y for the second interval is computed in a similar manner by using the formulae given above.

Problem:

1. Find the values of y (1.1) using fourth order Runge-Kutta method, given that $dy/dx = y^2 + xy$ and $y(1) = 1$

Solution:

Given $y' = f(x, y) = y^2 + xy$

Here it is given that $x_0 = 1$, $y_0 = 1$ and let $h = 0.1$

Now $k_1 = h f(x_0, y_0)$

$= h(y_0^2 + x_0 y_0)$

$= (0.1)[1^2 + (1)(1)] \quad = (0.1)(2)$

Therefore **k_1 = 0.2**

$k_2 = h f(x_0 + h/2, y_0 + k_1/2)$

$k_2 = h[(y_0 + k_1/2)^2 + (x_0 + h/2)(y_0 + k_1/2)]$

$= (0.1)[(1 + 0.2/2)^2 + (1 + 0.1/2)(1 + 0.2/2)]$

$= (0.1)[(1.1)^2 + (1.05)(1.1)]$

= 0.2365

$k_3 = h f(x_0 + h/2, y_0 + k_2/2)$

$k_3 = h[(y_0 + k_2/2)^2 + (x_0 + h/2)(y_0 + k_2/2)]$

$= (0.1)[(1 + 0.2365/2)^2 + (1 + 0.1/2)(1 + 0.2365/2)]$

$= (0.1)[(1.11825)^2 + (1.05)(1.11825)]$

$= (0.1)[2.4246]$

= 0.24246

$k_4 = h f(x_0 + h, y_0 + k_3)$

$k_4 = h[(y_0 + k_3)^2 + (x_0 + h)(y_0 + k_3)]$

$= (0.1)[(1 + 0.24246)^2 + (1 + 0.1)(1 + 0.24246)]$

$= (0.1)[(1.24246)^2 + (1.01)(1.24246)]$

$= (0.1)[1.5437 + 1.366706]$

$= (0.1)[2.9104]$

= 0.29104

$\Delta y = 1/6(k_1 + 2k_2 + 2k_3 + k_4)$

$\Delta y = 1/6[0.2 + 2(0.2365) + 2(0.24246) + 0.29104]$

$= 1/6[1.44896]$

$= 0.24149$

Therefore $y_1 = y_0 + \Delta y$

$= 1 + 0.24149$

$= 1.24149$

y (1.1) = 1.24149

2. Solve the differential equation y' = y – x with the initial condition y(0.1) = 2.20517 and find y(0.2) by fourth order Runge-Kutta method.

Solution:

Given: $f(x,y) = y - x$ and $x_0 = 0.1$, $y_0 = 2.20517$ and let $h = 0.1$

Now $k_1 = h\, f(x_0, y_0) = 0.1(2.20517-0.1) = 0.210517$

$k_2 = h\, f(x_0 + h/2, y_0 + k_1/2) = h[\; y_0 + k_1/2 \; - \; (x_0 + h/2)]$

$= 0.1[\; (92.20517 + 0.2105/2) - (0.1 + 0.1/2)]$

$= 0.1[2.31042-0.15] = 0.21604$

$k_3 = h\, f(x_0 + h/2, y_0 + k_2/2)$

$= h[\; y_0 + k_2/2 \; - \; (x_0 + h/2)]$

$= 0.1[\,(2.20517 + 0.21604/2) - (0.1 + 0.1/2)\,]$

$= 0.1[2.31319- 0.15) = 0.21632$

$k_4 = h\, f(x_0 + h, y_0 + k_3)$

$k_4 = h[\; (y_0 + k_3) \; - \; (x_0 + h)\,]$

$= 0.1[\,(2.20517+ 0.21632) - (0.1 + 0.1)]$

$= 0.22214$

$\Delta y = 1/6(k_1 + 2k_2 + 2k_3 + k_4)$

$\Delta y = 1/6\,[\,0.2105 + 2(0.21604) + 2(0.21632) + 0.22214\,]$

$= 0.21622$

Therefore $y_1 = y_0 + \Delta y$

$= 2.20517 + 0.21622 = 2.42139$

$Y(0.2) = 2.42139$

****

MILNE'S PREDICTOR CORRECTOR METHOD

The function $y(x_0 + rh)$ can be expanded as

$$y(x_0 + rh) = y_0 + r\Delta y_0 + \frac{r(r-1)\Delta^2 y_0}{2!} + \frac{r(r-1)(r-2)\Delta^3 y_0}{3!} + \frac{r(r-1)(r-2)(r-3)\Delta^4 y_0}{4!} + \ldots$$

Differentiating w.r.t. 'r' we get

$$h\,y'(x_0 + rh) = \Delta y_0 + [(2r-1)/2]\,\Delta^2 y_0 + [(3r^2 - 6r + 2)/6]\,\Delta^3 y_0 + [(2r^3 - 9r^2 + 11r - 3)/12]\,\Delta^4 y_0 + \ldots \quad (1)$$

Putting $r = 1, 2, 3$ and 4 in (1) we get

$$hy'(x_0 + h) = \Delta y_0 + 1/2\,\Delta^2 y_0 - 1/6\,\Delta^3 y_0 + 1/12\,\Delta^4 y_0 + \ldots \quad (2)$$

$$hy'(x_0 + 2h) = \Delta y_0 + 3/2\,\Delta^2 y_0 + 1/3\,\Delta^3 y_0 - 1/12\,\Delta^4 y_0 + \ldots \quad (3)$$

$$hy'(x_0 + 3h) = \Delta y_0 + 5/2\,\Delta^2 y_0 + 11/6\,\Delta^3 y_0 + 1/4\,\Delta^4 y_0 + \ldots \quad (4)$$

$$hy'(x_0 + 4h) = \Delta y_0 + 7/2\,\Delta^2 y_0 + 13/3\,\Delta^3 y_0 + 25/12\,\Delta^4 y_0 + \ldots \quad (5)$$

Now

$$\left.\begin{aligned} y(x_0 + h) &= y_1 \Rightarrow y'(x_0 + h) = y_1' \\ y(x_0 + 2h) &= y_2 \Rightarrow y'(x_0 + 2h) = y_2' \\ y(x_0 + 3h) &= y_3 \Rightarrow y'(x_0 + 3h) = y_3' \\ y(x_0 + 4h) &= y_4 \Rightarrow y'(x_0 + 4h) = y_4' \end{aligned}\right\} \quad \ldots (6)$$

Substituting (6) in (2), (3), (4) and (5)

$$y_1' = 1/h\,[\Delta y_0 + 1/2\,\Delta^2 y_0 - 1/6\,\Delta^3 y_0 + 1/12\,\Delta^4 y_0 + \ldots] \ldots (7)$$

$$y_2' = 1/h\,[\Delta y_0 + 3/2\,\Delta^2 y_0 + 1/3\,\Delta^3 y_0 - 1/12\,\Delta^4 y_0 + \ldots] \ldots (8)$$

$$y_3' = 1/h\,[\Delta y_0 + 5/2\,\Delta^2 y_0 + 11/6\,\Delta^3 y_0 + 1/4\,\Delta^4 y_0 + \ldots] \ldots (9)$$

$$y_4' = 1/h\,[\Delta y_0 + 7/2\,\Delta^2 y_0 + 13/3\,\Delta^3 y_0 + 25/12\,\Delta^4 y_0 + \ldots] \ldots (10)$$

We know that

$$\left.\begin{aligned} \Delta y_0 &= y_1 - y_0 \\ \Delta^2 y_0 &= y_2 - 2y_1 + y_0 \\ \Delta^3 y_0 &= y_3 - 3y_2 + 3y_1 - y_0 \end{aligned}\right\} \quad \ldots (11)$$

$\Delta^4 y_0 \quad = \quad y_4 - 4y_3 + 6y_2 - 4y_1 + y_0$

Neglecting differences beyond the fourth power in (7) , (8), (9) and (10) and replacing the remaining differences by (11) we get

$$y_1' = 1/h\,[\,(y_1\ y_o) + 1/2\,(y_2\ -2y_1 + y_0) - 1/6\,(y_3 - 3y_2 + 3y_1 - y_0$$

$$+ 1/12\,(y_4 - 4y_3 + 6y_2 - 4y_1 + y_0)\,]$$

$$y_1' = 1/12\,h\,[\,-3y_0 - 10y_1 + 18y_2 - 6y_3 + y_4\,] \qquad \ldots (12)$$

Similarly we get

$$y_2' = 1/12\,h\,[\,y_0 - 8y_1 + 8y_3 - y_4\,] \qquad \ldots (13)$$

$$y_3' = 1/12\,h\,[\,-y_0 + 6y_1 - 18y_2 + 10y_3 + 3y_4\,] \qquad \ldots (14)$$

$$y_4' = 1/12\,h\,[\,3y_0 - 16y_1 + 36y_2 - 48y_3 + 25y_4\,] \qquad \ldots (15)$$

Now our aim is to find y_4

$$2\,(y_1' + y_3') = 2/12h\,[\,-4\,y_0 - 4\,y_1 + 4\,y_3 + 4\,y_4\,] \qquad \ldots (16)$$

$$y_2' = 1/12h\,[\,y_0 - 8\,y_1 + 8\,y_3 - y_4\,] \qquad \ldots (17)$$

(16) – (17) => $\quad = \quad 2\,y_1' + 2\,y_3' - y_2'$

$$= 1/12h\,[\,-9\,y_0 + 9\,y_4\,]$$

$$= 9/12h\,[\,y_4 - y_0\,]$$

$$y_4 - y_0 = 4h/3\,[\,2y_1' - y_2' + 2\,y_3'\,]$$

(or) $\quad y_4 = y_0 + 4h/3\,[\,2y_1' - y_2' + 2\,y_3'\,] \qquad \ldots (18)$

In general we can write (18) as

$$y_{n+1} = y_{n-3} + 4h/3\,[\,2y_{n-2}' - y_{n-1}' + 2\,y_n'\,]$$

This is **Milne's Predictor formula** and is denoted by

$$\mathbf{y_{n+1,\,p} = y_{n-3} + 4h/3\,[\,2y_{n-2}' - y_{n-1}' + 2\,y_n'\,]} \qquad \ldots \mathbf{(19)}$$

This formula is in general compatible for the step-by-step solution of $y' = f(x, y)$. But, as a precaution against errors of various kinds, it is desirable to have a second, independent formula into which y_{n+1} can be substituted as a check. This formula is called **Milne's corrector formula** and can be obtained as follows.

$y_2{}' + y_4{}' \quad = \quad 1/12h\,[\,4\,y_0 - 24\,y_1 + 36\,y_2 - 40\,y_3 + 24\,y_4\,]$ … (20)

$4\,y_3{}' \quad = \quad 4/12h\,[\,-\,y_0 + 6\,y_1 - 18\,y_2 + 10\,y_3 + 3\,y_4\,]$ … (21)

(20) + (21) =>

$$y_2{}' + 4\,y_3{}' + y_4{}' = 1/12h\,[\,36\,y_4 - 36\,y_2\,]$$

$$= 3h\,[\,y_4 - y_2\,]$$

i.e., $y_4 - y_2 = h/3\,[y_2{}' + 4\,y_3{}' + y_4{}'\,]$

$$\mathbf{y_4 = y_2 + h/3\,[y_2{}' + 4\,y_3{}' + y_4{}']}$$

This formula is called **Milne's corrector formula** and can be written in general as

$$\mathbf{y_{n+1,\,c} = y_{n-1} + h/3\,[y_{n-1}{}' + 4\,y_n{}' + y_{n+1}{}']}$$

If the initial four values are not given, we can obtain these values either by using Taylor series method or by Runge-Kutta method.

Problem:

1. The differential equation $dy/dx = y - x^2$ is satisfied by $y\,(0) = 1$, $y\,(0.2) = 1.12186$, $y\,(0.4) = 1.46820$, $y\,(0.6) = 1.7379$. Compute the value of $y\,(0.8)$ by Milne's Predictor – Corrector Method.

Solution:

Given $dy/dx = y' = y - x^2$ and $h = 0.2$

$x_0 = 0$		$y_0 = 1$	
$x_1 = 0.2$		$y_1 = 1.12186$	
$x_2 = 0.4$		$y_2 = 1.46820$	
$x_3 = 0.6$		$y_3 = 1.7379$	
$x_4 = 0.8$		$y_4 = ?$	

By **Milne"s Predictor formula** we have

$$y_{n+1,\,p} = y_{n-3} + 4h/3\,[\,2y_{n-2}{}' - y_{n-1}{}' + 2\,y_n{}'\,] \quad \text{… (1)}$$

To get y_4, put $n = 3$ in (1) we get

$y_{4,p}$ = $y_0 + 4h/3\,[\,2y_1' - y_2' + 2\,y_3'\,]$ … (2)

Now y_1' = $(y - x^2)_1$ = $y_1 - x_1^2$

= $1.12186 - (0.2)^2$ = 1.08186 … (3)

Y_2' = $(y - x^2)_2$ = $y_2 - x_2^2$

= $1.46820 - (0.4)^2$ = 1.3082 … (4)

y_3' = $(y - x^2)_3$ = $y_3 - x_3^2$

= $1.7379 - (0.6)^2$ = 1.3779 … (5)

Substituting (3) , (4) and (5) in (2) we get

$y_{4,p}$ = 1 + 4 (0.2)/3 [2 (1.08186) – 1.3082 + 2 (1.3779)]

= 1 + 0.266 [2.1637 – 1.3082 + 2.7558]

= 1.9630187

Therefore y (0.8) = 1.9630187 (By Milne's Predictor Formula)

By Milne's Corrector Formula we have

$y_{n+1,c}$ = $y_{n-1} + h/3\,[y_{n-1}' + 4\,y_n' + y_{n+1}']$

To get y_4 , put n = 3 , we get

$y_{4,c}$ = $y_2 + h/3\,[y_2' + 4\,y_3' + y_4']$ … (6)

Now y_4' = $(y - x^2)_4$ = $y_4 - x_4^2$

= $1.96301 - (0.8)^2$

= 1.3230187 … (7)

Substituting (4) , (5) and (7) in (6) we get

$y_{4,c}$ = 1.46820+(0.2)/3[1.3082 + 4 (1.3779) + 1.3230187]

= 2.0110546

y (0.8) = 2.0110546 (By Milne's Corrector Formula)

Problem

2. Using Taylors series method, solve $dy/dx = xy + y^2$, $y(0) = 1$ at $x = 0.1$, 0.2 and 0.3 continue the solution at $x = 0.4$ by Milne's predictor corrector method.

Solution:

Given $y' = xy + y^2$ and $x_0 = 0$ $y_0 = 1$ and $h = 0.1$

Now
$$y' = xy + y^2$$
$$y'' = xy' + y + 2yy'$$
$$y''' = xy'' + 2y' + 2yy'' + 2y^{2'}$$

Since the values of y's are not given directly we can find them by using Taylors method as given below

To find y (0.1)

By Taylors series we have

$$y(0.1) = y_1 = y_0 + hy_0' + (h^2/2!)\, y_0'' + (h_3/3!)\, y_0''' + \dots \qquad \dots (1)$$

$$y_0' = (xy + y^2)_0 = (x_0y_0 + y_0^2) = 1 \qquad \dots (2)$$

$$y_0'' = (xy' + y + 2yy')_0$$

$$= (x_0y_0' + y_0 + 2y_0y_0') = 3 \qquad \dots (3)$$

$$y_0''' = (xy'' + 2y' + 2yy' + 2y^{2'})_0 = 10 \qquad \dots (4)$$

Substituting (2), (3) and (4) in (1) we get

$$y(0.1) = 1 + (0.1) + [(0.1)^2/2]\, 3 + [(0.1)^3/6]\, 10$$

$$= 1 + 0.1 + 0.015 + 0.001666$$

Therefore y (0.1) = 1.11666

To find y (0.2)

By Taylors series we have

$$y_2 = y_1 + hy_1' + (h^2/2!)\, y_1'' + (h_3/3!)\, y_1''' + \dots \qquad \dots (5)$$

Now
$$y_1' = (xy + y^2) = x_1y_1 + y_1^2$$

$$= (0.1)(1.11666) + (1.11666)^2$$

$= \quad 0.111666 + 1.2469$

$= \quad 1.3585 \qquad \ldots (6)$

$y_1'' \quad = \quad (xy' + y + 2\, yy')$

$= \quad x_1 y_1' + y_1 + 2\, y_1 y_1'$

$= \quad (0.1)(1.3585) + 1.11666 + 2 (1.11666)(1.3585)$

$= \quad 0.13585 + 1.11666 + 3.0339$

$= \quad 4.2865 \qquad \ldots (7)$

$y_1''' \quad = \quad (xy'' + 2\, y' + 2\, yy'' + 2\, y^{2\prime})$

$= \quad (x_1 y_1'' + 2\, y_1' + 2\, y_1 y_1'' + 2\, y_1^{2\prime})$

$= \quad (0.1)(4.2865) + 2 (1.3585) + 2 (1.1167)(4.2865) + 2 (1.3585)^2$

$= \quad 0.4287 + 2.717 + 9.5735 + 3.6910$

$= \quad 16.4102 \qquad \ldots (8)$

Substituting (6) , (7) and (8) in (5) we get

$y (0.2) \quad = \quad 1.1167 + (0.1)(1.3585) + [(0.1)^2/2)(4.2865)] + [(0.1)^3/6)(16.4102)]$

$= \quad 1.1167 + 0.13585 + 0.0214 + 0.002735$

$= \quad 1.27668$

Therefore y (0.2) = 1.27668

To find y (0.3)

By Taylors series we have

$Y_3 = y_2 + hy_2' + (h^2/2!)\, y_2'' + (h_3/3!)\, y_2''' + \ldots \qquad \ldots (9)$

Now $y_2' \quad = \quad (xy + y^2)_2 \quad = \quad x_2 y_2 + y_2^2$

$= \quad (0.2) (1.2767) + (1.2767)^2$

$= \quad 0.2553 + 1.6299$

$= \quad 1.8852 \qquad \ldots (10)$

y_2^{II} = $(xy^I + y + 2\ yy^I)_2$

= $x_2y_2^I + y_2 + 2\ y_2y_2^I$

= (0.2) (1.8852) + 1.2767 + 2 (1.2767)(1.8852)

= 0.3770 + 1.2767 + 4.8136

= 6.4674 … (11)

y_1^{III} = $(xy^{II} + 2\ y^I + 2\ yy^{II} + 2\ y^{2I})_2$

= $(x_2y_2^{II} + 2\ y_2^I + 2\ y_2y_2^{II} + 2\ y_2^{2I})$

= (0.2)(6.4674) + 2 (1.8852) + 2 (1.2767)(6.4674) + 2 $(1.8852)^2$

= 1.2934 + 3.7704 + 16.5139 + 7.1079

= 28.6855 … (12)

Substituting (10) , (11) and (12) in (9) we get

y (0.3) = 1.2767 + 2 (1.8852) + [$(0.1)^2/2)(6.4674)$] + [$(0.1)^3/6)(28.6855)$]

= 1.2767 + 0.18852 + 0.0323 + 0.004780

= 1.5023

Therefore y (0.3) = 1.5023

Therefore we have the following values

x_0	=	0	y_0	=	1
x_1	=	0.1	y_1	=	1.11666
x_2	=	0.2	y_2	=	1.27668
x_3	=	0.3	y_3	=	1.5023

To find y (0.4) by **Milne"s Predictor formula**

y_{n+1}, p = $y_{n-3} + 4h/3\ [\ 2y_{n-2}^I - y_{n-1}^I + 2\ y_n^I\ ]$ … (13)

y_3^I = $(xy + y^2)_3$

= $x_3y_3 + y_3^2$

$= [(0.3) (1.5023) + (1.5023)^2]$

$= 0.45069 + 2.2569$

$= 2.7076$

Putting n = 3 in (13) we get

$y_4, p = y_0 + 4h/3 [2y_1' – y_2' + 2 y_3']$

$y_4, p = 1 + 4 (0.1)/3 [2 (1.3585) – 1.8852 + 2 (2.7076)]$

$= 1 + 0.1333 [2.717 – 1.8852+ 5.4152]$

$= 1.8329$

Therefore y_4, p = 1.8329 (By Milne's Predictor Formula)

To find y = (0.4) by Milne's Corrector Formula:

By Milne's Corrector Formula we have

$y_{n+1}, c = y_{n-1} + h/3 [y_{n-1}' + 4 y_n' + y_{n+1}']$ … (14)

Now $y_4' = (xy + y^2)_4 = x_4y_4 + y_4^2$

$= [(0.4)(1.8329) + 1.8329)^2]$

$= 0.7332 + 3.3595$

$= 4.0927$

,Putting n = 3 , in (14) we get

$y_4, c = y_2 + h/3 [y_2' + 4 y_3' + y_4']$

$y_4, c = 1.27668 + (0.1)/3 [1.8852 + 4 (2.7076) + 4.0927]$

$= 1.27668 + 0.0333 [1.8852 + 10.8304 + 4.0927]$

$= 1.8369$

The value of y(0.4) calculated by Milne's Corrector Formula is

y (0.4) = 1.8369

ADAM'S MOULTON METHOD:

Let $dy/dx = f(x,y)$ be the differential equation to be solved with the i9nitial condition $y_0 = y(x_0)$. we have to compute

$$y_{-1} = y(x_0 - h)$$

$$y_{-2} = y(x_0 - 2h) \quad \text{and}$$

$$y_{-3} = y(x_0 - 3h)$$ by Taylor series or Runge-Kutta Method.

Then calculate

$$f_{-1} = f(x_0 - h, y_{-1})$$

$$f_{-2} = f(x_0 - 2h, y_{-2})$$

$$f_{-3} = f(x_0 - 3h, y_{-3})$$

$$y_1 = y_0 + \oint_{x0}^{x1} f(x,y)dx$$

Using Newton's backward interpolation formula

$$f(x, y) = f_0 + n\nabla f_0 + \frac{n(n+1)}{2}\nabla^2 f_0 + \frac{n(n+1)(n+2)}{6}\nabla^3 f_0 + \ldots$$

We have

$$y_1 = y_0 + \oint_{x0}^{x1} f0 + n\nabla f0 + \frac{n(n+1)}{2}\nabla 2 f0 + \ldots)dx$$

Since $x = x_0 + nh$ $\quad dx = hdn$

$$y_1 = y_0 + \oint_0^1 f0 + n\nabla f0 + \frac{n(n+1)}{2}\nabla 2 f0 + \ldots)dx$$

$$y_1 = y_0 + h[f_0 + 1/2\nabla f_0 + 5/12\nabla^2 f_0 + 3/8\nabla^3 f_0 + \ldots]$$

Substituting for $\nabla f_0, \nabla^2 f_0 \ldots$ we get

$$\mathbf{y_1 = y_1^{(P)} = y_0 + h/24[55 f_0 - 59 f_{-1} + 37 f_{-2} - 9 f_{-3}]}$$

Here y_1^P is the Predictor formula.

To derive the Adam's Moulton Corrector formula, we have to use Newton's backward formula at f_1 i.e.,

$$f(x, y) = f_1 + n \nabla f_1 + \frac{n(n+1)}{2} \nabla^2 f_1 + \frac{n(n+1)(n+2)}{6} \nabla^3 f_1 + \ldots$$

$$y_1 = y_0 + \int_{x0}^{x1} \left(f1 + n \nabla f1 + \frac{n(n+1)}{2} \nabla 1\, f0 + \ldots \right) dx$$

Since $x = x_1 + nh$ $dx = hdn$

$$y_1 = y_0 + \int_0^1 \left(f1 + n \nabla f1 + \frac{n(n+1)}{2} \nabla 2\, f1 + \ldots \right) dx$$

$$y_1 = y_0 + h [f_1 - 1/2 \nabla f_1 - 1/12 \nabla^2 f_1 - 1/24 \nabla^3 f_1 + \ldots]$$

Substituting for $\nabla f_1 , \nabla^2 f_1 \ldots$ we get

$$y_1 = \mathbf{y_1^{(C)}} = \mathbf{y_0 + h/24 [9 f_1 + 19 f_0 - 5 f_1 + f_{-2}]}$$

This is **Adam's – Moulton Corrector Formula**

Problem:

1. Solve the equation $dy/dx = x^2 (1+y)$ with $y(1) = 1$ $y(1.1) = 1.233$ $y(1.2) = 1.548$ and $y(1.3) = 1.979$ evaluate $y(1.4)$ by Adam's Moulton Method.

Solution:

Given $f(x,y) = x^2 (1 + y)$

$h = 0.1$

When $x = 1$ $y = 1$

Therefore $f_{-3} = f(x,y)$

$= (1)^2 (1 + 1)$

$= 2$

When $x = 1.1$ $y = 1.233$

Therefore $f_{-2} = f(x,y)$

$= (1.1)^2 (1.1 + 1.233)$

		=	2.702
When	x	=	1.2 y = 1.548
Therefore	f_{-1}	=	$f(x,y)$
		=	$(1.2)^2 (1.2 + 1.548)$
		=	3.669
When	x	=	1.3 y = 1.979
Therefore	f_0	=	$f(x,y)$
		=	$(1.3)^2 (1.3 + 1.979)$
		=	5.035

The Predictor formula is

$$y_1^{(P)} = y_0 + h/24 [55 f_0 - 59 f_{-1} + 37 f_{-2} - 9 f_{-3}]$$

$$= 2.4011 + (0.1/24) [(55 \times 5.035) - (59 \times 3.669) + (37 \times 2.702) - (9 \times 2)]$$

$$= 2.573$$

The Corrector formula is

$$y_1^{(c)} = y_0 + h/24 [9 f_1 + 19 f_0 - 5 f_{-1} + f_{-2}]$$

$$f_1 = f(x_1, y_1)$$

$$= x^2 (1 + y)) \quad \text{when} \quad x = 1.4, \quad y = 2.573$$

$$= (1.4)^2 (1.4 \times 2.573)$$

$$= 7.004$$

Therefore $y_1^{(c)} = 1.979 + (0.1/24) [(9 \times 7.004) + (19 \times 5.035) - (5 \times 3.609) + (2.702)]$

$$= 2.575$$

Therefore $y(1.4) = 2.575$

2. Find $y(0.4)$ given that $y' = 1 + xy$ and $y(0) = 2$, $y(0.1) = 2.1103$, $y(0.2) = 2.243$ and $y(0.3) = 2.4011$ by Adam's Moulton Predictor-Corrector method.

Solution:

Given $f(x,y) = 1 + xy$

$h = 0.1$

$f_{-3} = f(x,y)$ when $x = 0, y = 2$

$= 1 + (0 \times 2)$

$= 1$

$f_{-2} = f(x_{-2}, y_{-2})$ when $x = 0.1 \quad y = 2.1103$

$= 1 + (0.1 \times 2.1103)$

$= 1.21103$

$f_{-1} = f(x_{-1}, y_{-1})$ when $x = 0.2 \quad y = 2.243$

$= 1 + (0.2 \times 2.243)$

$= 1.4486$

$f_0 = f(x_0, y_0)$ when $x = 0.3 \quad y = 2.4011$

$= 1 + (0.3 \times 2.4011)$

$= 1.7203$

The Predictor formula is

$y_1^{(P)} = y_0 + h/24 [55 f_0 – 59 f_{-1} + 37 f_{-2} – 9 f_{-3}]$

$= 2.4011 + (0.1/24) [(55 \times 1.72033) – (59 \times 1.4486) + (37 \times 1.21103) – (9 \times 1)]$

$= 2.5884$

The Corrector formula is

$y_1^{(c)} = y_0 + h/24 [9 f_1 + 19 f_0 - 5 f_{-1} + f_{-2}]$

$f_1 = f(x_1, y_1)$

= (1 + xy) when $x = x_1$, $y = y_1$

= (1 + xy) when x = 0.4, y = 2.5884

= 1+ (0.4 x 2.5884)

= 2.0354

Therefore $y_1^{(c)}$ = 2.4011 + (0.1/24) [(9 x 2.0354) + (19 x 1.72033)

- (5 x 1.4486) + (1.21103)]

= 2.5885

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